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On compact complex affine manifolds

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Introduction.

In this paper we study compact complex affine manifolds. Let $A(n, \boldsymbol{C})$ be the group of the affine transformations on $\boldsymbol{C}^n$ and let Γ be a subgroup of $A(n, \boldsymbol{C})$ such that 1) Γ acts on $\boldsymbol{C}^n$ properly discontinuously and freely 2) $\boldsymbol{C}^n/\Gamma$ is compact. A compact complex manifold $\boldsymbol{C}^n/\Gamma$ is called a compact complex affine manifold. For $n=2$, such manifolds have been classified by Vitter [**6**], Fillmore and Scheuneman [**2**] and Suwa [**5**]. The purpose of this paper is to study the complex manifold $\boldsymbol{C}^n/\Gamma$ under certain conditions. Put

$$N(n, \boldsymbol{C})=\left\{A\in A(n, \boldsymbol{C})\ \middle|\ A=\begin{pmatrix} a & \alpha \\ 0 & 1\end{pmatrix},\ a=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix},\ \alpha\in\boldsymbol{C}^n\right\}.$$

In section 1 we show that if Γ is contained in $N(n, \boldsymbol{C})$, then every non-zero holomorphic vector field on $\boldsymbol{C}^n/\Gamma$ has no zero point and the Lie algebra $\mathfrak{a}$ of all holomorphic vector fields on $\boldsymbol{C}^n/\Gamma$ is solvable and of dimension $\leqq n$. In section 2 we study the case when Γ is contained in $N(n, \boldsymbol{C})$ and the Lie algebra $\mathfrak{a}$ is of n-dimension. In this case we show that there exist a simply connected complex nilpotent Lie subgroup G of $N(n, \boldsymbol{C})$ which contains Γ and a biholomorphic map $\phi:\boldsymbol{C}^n\to G$ such that $\phi(\gamma(z))=\gamma\phi(z)$ for any $\gamma\in\Gamma$ and any $z\in\boldsymbol{C}^n$. In particular, we see that there is a biholomorphic map $\bar{\phi}:\boldsymbol{C}^n/\Gamma\to\Gamma\backslash G$. In section 3 we show that if Γ is contained in $N(n, \boldsymbol{C})$ and $\boldsymbol{C}^n/\Gamma$ has a Kähler metric, then $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus. In section 4 we consider the case when Γ is an abelian subgroup of $A(n, \boldsymbol{C})$ and prove that $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus.

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§ 1. Preliminaries.

Let $A(n, \boldsymbol{C})$ be the group of all affine transformations on $\boldsymbol{C}^n$. The group $A(n, \boldsymbol{C})$ is represented by the group of all matrices of the form $A=\begin{pmatrix} a & \alpha \\ 0 & 1\end{pmatrix}$ where $a=(a_{ij})\in GL(n, \boldsymbol{C})$ and $\alpha=(\alpha_i)\in\boldsymbol{C}^n$ is a column vector. Let $N(n, \boldsymbol{C})$