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On the signature invariants of a non-singular complex sesqui-linear form

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The purpose of this note is to make clear the relationship between two types of signatures defined for a non-singular real bilinear or complex sesqui-linear form, and then, to get a result in the algebraic topology.

Let $l: V\times V\rightarrow \boldsymbol{C}$ be a complex sesqui-linear form of finite dimension; a matrix representation $x^*\Gamma y$ is used and a symbol "$*$" stands for the transpose of the conjugate of the matrix or the vector. Let t be an indeterminant which may be thought either as an automorphism or as a variable ranging over the complex numbers. We call $\Gamma(t)=\Gamma-\Gamma^*t$ an Alexander matrix and $\det\Gamma(t)$ the Alexander polynomial. The first series of signatures consists of the signature τ_ω of the hermitian form $l_\omega=x^*\Gamma_\omega y$ with $\Gamma_\omega=(1/2)\{(1-\bar{\omega})\Gamma+(1-\omega)\Gamma^*\}$. Since $\tau_\xi=\mathrm{sign}(1-\mathrm{Re}\,\xi)\tau_\omega$ with $\omega=-(1-\xi)/(1-\bar{\xi})$, the only interesting case is when ω is on the unit circle, where Γ_ω reduces to $\Gamma_\omega=(1/2)(1-\bar{\omega})\Gamma(\omega)$.

A hermitian form $l_+=x^*Ay$ where $A=(1/2)(\Gamma+\Gamma^*)$ and a skew-hermitian form $l_-=x^*(-Q)y$ where $Q=(1/2)(\Gamma^*-\Gamma)$ are considered; then $\Gamma=A-Q$ and of course $2A=\Gamma_{-1}$. If the form l is non-singular, then the matrix $P=(\Gamma^*)^{-1}\Gamma$ gives an automorphism t of l, i.e., $P^*\Gamma P=\Gamma$, and hence of l_ω, l_+ and l_-. The eigen-values α of the automorphism t associate another series of signatures $\sigma_{(\alpha)}$ which are defined by the hermitian form l_+; where l_+ is restricted to the α-root subspaces $V_\alpha=\{x\in V; (t-\alpha)^k x=0$ for some $k\}$. Note that $\dim V_\alpha>0$ if and only if α is a root of the Alexander polynomial and we have a generalized Cayley transformation $Q(I+P)=A(I-P)$. Moreover, we can remark that, if $\alpha\neq\pm1$, $\sigma_{(\alpha)}=\mathrm{sign}(V_\alpha; l_+)$ is equal to $\mathrm{sign}(\mathrm{Im}\,\alpha)\,\mathrm{sign}(V_\alpha; il_-)$. (Cf. § 1, case (b).) We define $\sigma_{(-1\pm0i)}$ by $\pm\mathrm{sign}(V_{-1}; il_-)$.

THEOREM 1 (Complex case). *For* $\omega=\exp(i\varphi)$ *and* $\alpha=\exp(i\theta)$ *with* $-\pi<\varphi<\pi$ *and* $-\pi<\theta<\pi$,

$$(*)\qquad \tau_\omega=\mathrm{sign}(\mathrm{Im}\,\omega)\{\sum_{|\alpha|=1,\alpha\neq-1}\mathrm{sign}(\varphi-\theta)\sigma_{(\alpha)}+\sigma_{(-1+0i)}\}$$

holds, provided either the automorphism t is semi-simple, or ω is not a root of the Alexander polynomial.

REMARK. If $\omega=-1$, (*) is replaced by (*′) $\mathrm{sign}(l_+)=\sum\sigma_{(\alpha)}(|\alpha|=1, \alpha\neq-1)$. The formula, (*) or (*′), does not always hold. The excluded cases will be