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Freeness of the group $\langle a^n, b^n \rangle$ for some integer n, $a, b \in SL(2, C)$

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Suppose that $\langle a, b\rangle$ is an irreducible subgroup of the real special linear group $SL(2, R)$ with $\mathrm{tr}\, a=\alpha$, $\mathrm{tr}\, b=\beta$, $\mathrm{tr}\, ab=\gamma$. Let $\alpha \geqq 2$, $\beta \geqq 2$. Purzitsky [7] and Rosenberger [9] proved that $\gamma \geqq \alpha\beta+2$ or $\gamma \leqq -2$ are the necessary and sufficient conditions for $\langle a, b\rangle$ to be the discrete free product of cyclic group $\langle a\rangle$ and $\langle b\rangle$. For $\alpha, \beta \in C$, suppose that $|\alpha| \geqq 2 |\beta| \geqq 2$ and $\langle a, b\rangle$ is not a free product of $\langle a\rangle$ and $\langle b\rangle$. What can we say about the freeness of the groups $\langle a^n, b^n\rangle$ for some integer n? In the present paper we shall discuss this question.

It was shown (cf. [5] Theorem 3.5) that if $\mathrm{tr}\, a=2=\mathrm{tr}\, b$ then a, b can be reduced simultaneously into the form:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \gamma-2 \\ 0 & 1 \end{pmatrix} \qquad \gamma \neq 2$$

respectively. In this case, a positive integer n can be chosen such that

$$|n(\gamma-2)| \geqq 4$$

so that $\langle a^n, b\rangle$ is free by a result of Chang, Jennings and Ree [1], even though $\langle a, b\rangle$ need not be free. Consequently $\langle a^n, b^n\rangle$ is free for some integer sufficiently large. However if the traces of both a and b are not equal to 2, then it is not so obvious that we conclude about the freeness of $\langle a^n, b^n\rangle$. We shall show that if $|\alpha|>2$, $|\beta|>2$, and $\langle a, b\rangle$ is irreducible then there always exists an integer n such that $\langle a^n, b^n\rangle$ becomes a free group. We shall prove that if the trace of one of the a and b is 2 while that of the other is >2, and a, b are non-trivial elements in $SL(2, R)$, then $\langle a^n, b^n\rangle$ is free for sufficiently large n. Throughout this paper, R and C stand for the sets of real and complex numbers respectively. I denotes the 2×2 identity matrix. Explanation for other concepts can be found in Dixon [2] or Wehrfritz [10].

Before we prove our main theorems, we mention some of the results used to prove them.

1. PING PONG LEMMA OF MACBEATH [4]. *Let A and B be groups of permutations of a set Ω and let G be the group generated by A and B together.*