On a parametrix in some weak sense of a first order linear partial differential operator with two independent variables

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Introduction.

Let $P(x, D)$ be a partial differential operator of order $m$ with $C^\infty$ coefficients in an open subset $\Omega$ of $\mathbb{R}^n$. According to Trèves [6], a local subelliptic estimate for $P$ near a point $x_0$ of $\Omega$ is an estimate of the form

$$
\|u\|_{m-1+\delta} \leq C \|Pu\|_0 \quad \text{for all } u \in C^\infty_0(U),
$$

where $\delta$ is a number such that $0 < \delta \leq 1$, $C$ is a positive constant, and $U$ is an open neighborhood of $x_0$ in $\Omega$.

In [6] Trèves established (0.1) for $\delta = \frac{1}{2k+1}$ when $P$ is an operator of principal type in $\Omega$ having the property:

$$
\text{(0.2) } \text{Let } P_m(x, D) \text{ be the principal part of } P. \text{ For any } (x_0, \xi^0) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \text{ and any complex number } z \text{ such that } P_m(x_0, \xi^0) = 0, d_\xi \text{Re}(zP_m)(x_0, \xi^{0}) \neq 0, \text{ the function } \text{Im}(zP_m)(x, \xi), \text{ restricted to the bicharacteristic strip of } \text{Re}(zP_m)(x, \xi) \text{ through } (x_0, \xi^0), \text{ has only zeros of even order less than or equal to } 2k.
$$

He reduced the proof of (0.1) to the estimate

$$
\|u\|_{\frac{1}{2k+1}, s} + \|D_t u\|_0 \leq C \|D_t u - i\beta(x, t, D_x)u\|_0, \quad u \in C^\infty_0(U),
$$

where $\|u\|_{r,s} = \int\int(1+|\xi|^r)(1+\tau^s)|\hat{u}(\xi, \tau)||^{r}d\xi d\tau$ for every real numbers $r,s,C$ is a positive constant, $\beta(x, t, D_x)$ is a first order pseudo-differential operator defined in an open neighborhood of 0 in $\mathbb{R}^{n+1}$ having the following property (0.4), and $U$ is an appropriate open neighborhood of 0.

$$
\text{(0.4) There is an open neighborhood } W \text{ of } 0 \text{ in } \mathbb{R}^n \text{ and a number } t_0 > 0 \text{ such that, for every } (x, \xi) \in W \times (\mathbb{R}^n \setminus \{0\}) \text{ } \beta(x, t, \xi) \text{ as a function of } t, |t| < t_0, \text{ has only zeros of even order less than or equal to } 2k.\n$$