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The representability of modular forms by theta series

By Arnold PIZER*

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§ 1. Introduction.

Let N be a natural number and $\Gamma_0(N)$ the congruence modular group of level N, i.e. $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, Z)\mid c\equiv 0 \pmod{N}\right\}$. Let $S_k(N)$ denote the space of cusp forms (of Haupt or principal type) on $\Gamma_0(N)$ of weight k, $k\geqq 2$. Eichler considered the problem ("the basis problem") of concretely giving a basis of $S_k(N)$ in terms of theta series attached to a rational quaternion algebra in the case N is square free (see [2], [3] and [4]). Hijikata and Saito ([6]) generalized Eichler's results to the case N has at least one simple prime factor (i.e. $N=pM$, $(p, M)=1$). Here we generalize these results to the case N is not a perfect square.

The basic idea is to consider Brandt Matrices $B(n)$ which occur in the theory of quaternion algebras and which are analogous to the Hecke Operators $T(n)$. In fact they both generate semi-simple commutative rings and the Brandt Matrices give a representation of the Hecke Operators on a space generated by theta series. In [8] we defined more general Brandt Matrices than considered in [4] or [6] and also computed their traces. Theorem 4 below gives a relation between the trace of the Brandt Matrix and the traces of Hecke Operators. From this theorem we obtain several results on the representability of modular forms by theta series. In particular Corollary 7 gives an explicit procedure for obtaining (in a concrete manner) all new forms on $\Gamma_0(N)$ if N is not a perfect square and Corollary 11 gives a version of Eichler's main Theorem in Chapter IV of [4].

§ 2. Brandt matrices.

Let $q_1=p_1^{s_1}\cdots p_f^{s_f}$ where the p_i are distinct primes and the $f, s_1, \cdots, s_f$ are all *odd* positive integers. Let q_2 be any positive integer with $(q_1, q_2)=1$. Let $\mathfrak{A}$ be the (unique) quaternion algebra over Q ramified precisely at the primes $\{p_1, \cdots, p_f, \infty\}$. Let $\mathfrak{O}$ be an order of level q_1q_2 in $\mathfrak{A}$ (see Definition 22 in [8]). Fixing a set of representatives of the (left) $\mathfrak{O}$-ideal classes, we define (gener-

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