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On the values of ray-class L-functions for real quadratic fields

To the memory of Taira Honda

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Let K be a real quadratic field and $\mathfrak{f}$ an integral ideal in K. Let χ be a character of the ray-class group mod $\mathfrak{f}$. We consider only the cases

(i) $$\chi((\alpha))=\chi(\alpha), \qquad \alpha\in K$$

and

(ii) $$\chi((\alpha))=(N(\alpha)/|N(\alpha)|)\chi(\alpha), \qquad \alpha\in K,$$

where χ in the right means, respectively, the character of the residue class group mod $\mathfrak{f}$, attached to the χ in the left.

In the present paper, we shall give the explicit formulas for $L(2k, \chi)$ in the case (i) and $L(2k+1, \chi)$ in the case (ii), k being a positive integer. The formulas for them are already given by Siegel [7] in other shapes (and by the different method from us). Our formulas, different from Siegel's, express explicitly the role of the totally positive units congruent to 1 mod $\mathfrak{f}$.

In his paper [2], Barner gave the explicit formulas for values of the ring-class L-functions of certain types at integral arguments. His tools are the representation of L-functions by the integrals of Eisenstein series, given by Siegel [6], and the transformation formulas of certain Lambert series under modular substitutions given in Apostol [1], S. Iseki [3]. The main point of his computation is the use of certain differential operators which connect Eisenstein series with Lambert series. Here we follow, with some necessary changes and supplies, the method of Barner. In the special case $\mathfrak{f}=(1)$, our formula coincides with Barner's.

In the course of the computation, fundamental is the representation of L-functions by the integral of Eisenstein series. This is clarified by Siegel [6] and will be formulated in somewhat general point of view in our Appendix.

NOTATION. As usual, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ are fields of rational, real and complex numbers, respectively. $\boldsymbol{Z}$ is the ring of rational integers.

We denote by $\boldsymbol{N}$ the set of natural numbers. $M_n(R)$ is the total matrix ring of order n with coefficients in a ring R. $\boldsymbol{H}$ means the upper-half plane. For a matrix $Y=(y_{ij})$ in $M_n(\boldsymbol{R})$, with $y_{ij}>0$, we put