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# Two remarks on irreducible characters of finite general linear groups

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**Introduction.**

0-1. Let $k$ be a finite field and $K$ be a finite extension of $k$. It is well-known (and is easily verified) that any character of $K^{\times}=GL(1, K)$, invariant under the action of the Galois group of $K$ with respect to $k$, is a composition of the norm homomorphism from $K^{\times}$ onto $k^{\times}$ and a suitable character of $k^{\times}$. In this paper, we prove an analogous result for irreducible characters of finite general linear groups $GL_n(k)$. In more detail, let $\sigma$ be the Frobenius automorphism of $K$ with respect to $k$. Then $\sigma$ acts naturally on $GL_n(K)$ as an automorphism with the fixed points set $GL_n(k)$. An irreducible representation $R$ of $GL_n(K)$ is said to be *$\sigma$-invariant* if the representation $R^{\sigma}=R\circ\sigma$ is equivalent to $R$. If $R$ is $\sigma$-invariant, there exists a linear transformation $I_{\sigma}$ of the representation space $V$ of $R$ which satisfies

$$R(g)I_{\sigma}=I_{\sigma}R(g^{\sigma}) \qquad (\forall g\in GL_n(K))$$

($I_{\sigma}$ is unique up to a constant scalar factor). We extend any class function $\chi$ on $GL_n(k)$ to a class function on $GL_n(K)$ by setting

$$\chi(x)=\begin{cases}\chi(x') & \text{if there exists an } x'\in GL_n(k) \text{ which is} \\ & \text{conjugate to } x \text{ in } GL_n(K)\,, \\ 0 & \text{otherwise}\,.\end{cases}$$

This is possible since two elements in $GL_n(k)$ are conjugate if and only if they are conjugate in $GL_n(K)$.

Now, we have:

THEOREM 1. *Let notations be as above. For a suitable normalization of $I_{\sigma}$, there exists an irreducible character $\chi_R$ of $GL_n(k)$ which satisfies* trace $I_{\sigma}R(g)$ $=\chi_R(\mathrm{Norm}_{K/k}(g))$ $({}^{\forall}g\in GL_n(K))$, *where*

$$\mathrm{Norm}_{K/k}(g)=g^{\sigma^{m-1}}g^{\sigma^{m-2}}\cdots g^{\sigma}g \qquad (m=\deg K/k)\,.$$

*Moreover, the mapping $R\mapsto\chi_R$ establishes the bijection from the set of equivalence classes of $\sigma$-invariant irreducible representations of $GL_n(K)$ onto the set*