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Tightness of compact Hausdorff spaces and normality of product spaces

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Let X be a topological space. The *tightness* of X, denoted by $t(X)$, is less than or equal to the cardinal number τ if $M\subset X$ and $x\in Cl(M)$ imply the existence of $N\subset M$ such that the cardinality of N is $\leqq\tau$ and $x\in Cl(N)$. This important notion was introduced by A. V. Arhangel'skiĭ [**1**] and has been investigated by many mathematicians. Arhangel'skiĭ also gave a characterization of the tightness of compact spaces (see Lemma 4 below). In this paper we give another characterization. We show that the tightness of a compact space X is characterized by the normality of the product space of X with a space of ordinal numbers.

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§ 1. Preliminaries.

The symbol τ denotes an infinite cardinal number and τ^+ denotes the smallest cardinal number greater than τ. Ordinal numbers are denoted by α, β and γ. The initial ordinal number of τ is denoted by $\omega(\tau)$. Let X be a topological space. Then the set $\{x_\alpha : x_\alpha\in X$ and $\alpha<\omega(\tau)\}$ is called a free sequence in X of length τ if $Cl(\{x_\alpha : \alpha<\beta\})\cap Cl(\{x_\alpha : \alpha\geqq\beta\})=\emptyset$ for any $\beta<\omega(\tau)$. A space is called strongly τ-compact if the closure of any subset of cardinality $\leqq\tau$ is compact ([**4**], p. 762).

In the arguments below, all spaces are assumed to be completely regular and T_1.

LEMMA 1 ([**3**], Theorem 4). *Let X be a compact space and Y be a pseudo-compact space. Then $X\times Y$ is pseudo-compact.*

LEMMA 2 ([**3**], Theorem 1). *Let $X\times Y$ be pseudo-compact. Then $\beta(X\times Y)=\beta X\times\beta Y$, where βX is the Stone-Čech compactification of X.*

LEMMA 3. *Let M and N be disjoint closed sets in a normal space X. Then $Cl_{\beta X}(M)\cap Cl_{\beta X}(N)=\emptyset$.*

LEMMA 4 ([**2**], Theorem 1). *For any infinite compact space X,* $t(X)=\sup\{\tau :$ *there exists a free sequence in X of length τ*$\}$.

LEMMA 5 ([**5**], Theorem 1.4). *Let X be a paracompact space with $t(X)\leqq\tau$*