Nonlinear variational inequalities and fixed point theorems

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§ 1. Introduction.

It was proved by Hartman and Stampacchia [8] in 1966 that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping on a compact, convex subset $X$ of $\mathbb{R}^n$, then there exists $x_0 \in X$ such that $\langle Tx_0, x_0 - x \rangle \geq 0$ for all $x \in X$. This remarkable result has been investigated and generalized in various points of views by Browder [1], [2], Moré [10] and others. For example, Browder extended this theorem to the case of which our considering mappings $T$ are of a compact convex subset $X$ of a topological vector space $E$; see Theorem 2 of [2]. In § 2 of this paper, we shall obtain two generalizations of this Browder's theorem. One of them is Lemma 1 that has various applications. The other is Theorem 3 that generalizes the Browder's result to closed and convex sets in topological vector spaces. We shall also make use of Theorem 3 to prove Theorem 4 that generalizes Moré's theorem [10, Theorem 2.4]. In § 3, using Lemma 1, we shall prove some fixed point theorems. Theorem 5 and Theorem 9 extend Browder's fixed point theorems [1, Theorem 1], [2, Theorem 3]. In § 4, we shall discuss Sion's minimax theorem and Terkelsen's minimax theorem. At first, we shall show that Sion's theorem follows simply from the fundamental and useful theorem of Browder [2, Theorem 1]. Furthermore, we state a necessary and sufficient condition that a minimax condition holds. Using this, we shall generalize Terkelsen's minimax theorem; see Theorems 16 and 17. In § 5, we give another proof for Fan's theorem [5] concerning systems of convex inequalities. The proof is simple. Furthermore, using this Fan's result, we prove Fan's minimax theorem [4] and also obtain a generalization of the result of Browder [2, Lemma 1]; see Theorems 18, 19 and 20. At last, by Lemma 1, we generalize Browder's theorem [2, Theorem 6] for multi valued mappings; see Theorem 21. By the same methods, we shall also generalize Kakutani's fixed point theorem [9]; see Theorem 22.

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