J. Math. Soc. Japan
Vol. 28, No. 1, 1976

The ball coverings of manifolds

By Kazuaki KOBAYASHI and Yasuyuki TSUKUI.

(Received March 27, 1975)

§ 1. Introduction.

A compact manifold is covered by a finite number of balls. We can define the minimum number of such balls for any compact manifold. This minimum number, of course, is not only related with the dimension of a manifold but also strongly related with the structure of the manifold. Further it is not a homotopy invariant. The authors found out the *ball coverings* (defined in (1.1)) to be a useful tool in topology of manifolds, especially in low dimensions. In the literatures, the authors can find *weak ball coverings* (defined below) in Zeeman's result (§ 2, 2.2) [**12**] and in Glaser's examples (§ 3, 3.5) [**2**, **3**]. In the present paper we will make clear some of the usefulness of ball coverings in the PL category. Throughout this paper all manifolds are connected compact PL manifolds and maps are piecewise linear, unless otherwise stated. S^n and B^n mean always a PL n-sphere and a PL n-ball, respectively (in this paper a ball means a closed ball). $\cong$ and $\sim$ mean homeomorphic (or group isomorphic) and homologous, respectively. The closure of a set X is denoted by $Cl(X)$ and $\mathrm{Int}(X)$ and $\mathring{X}$ mean the interiour of X. ∂M is the boundary of a manifold M. $N(X; Y)$ is usually used for a regular neighborhood of X in Y. $\sharp A$ indicates the number of elements (or the number of connected components) of a set A.

1.1. DEFINITION. Let M^n be an n-manifold and $\mathcal{B}=\{B_i\}$ be a set of finite n-balls in M.

(1) $\mathcal{B}$ is called a *weak ball covering* of M if $\cup B_i=M$.

(2) $\mathcal{B}$ is called a *ball covering* of M if $\mathcal{B}$ is a weak one of M and $B_i\cap B_j=\partial B_i\cap\partial B_j$ is an $(n-1)$-manifold (may not be connected) for $B_i, B_j\in\mathcal{B}$ and $i\neq j$. Define

$$\beta(M)=\min.\{\sharp\mathcal{B}\mid\mathcal{B}\text{ is a weak ball covering of }M\}\text{ and}$$

$$b(M)=\min.\{\sharp\mathcal{B}\mid\mathcal{B}\text{ is a ball covering of }M\}.$$

We call a ball covering (or weak one) $\mathcal{B}$ of a manifold M to be minimal if $\sharp\mathcal{B}=b(M)$ (or $\beta(M)$, respectively). Obviously $\beta(M)\leqq b(M)$.

This paper consists of five sections.