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Fourier transforms on the motion groups

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§ 1. Introduction.

The purpose of the present paper is to characterize the images of some function spaces on the motion groups by the Fourier transform.

Let K be a connected compact Lie group acting on a finite dimensional real vector space V as a linear group. Let G be the semidirect product of V and K, i.e. G is the group comprised of all pairs (x, k) $(x \in V, k \in K)$ with the direct product topology, multiplication being given by $(x_1, k_1)(x_2, k_2) = (x_1 + k_1 x_2, k_1 k_2)$. G is called the motion group.

Let $\hat{V}$ be the dual space of V. For any $\xi \in \hat{V}$ we denote by U^{ξ} the induced representation of G by the unitary representation $x \mapsto e^{i<\xi, x>}$, $(i = \sqrt{-1})$ of the normal abelian subgroup V. U^{ξ} is not irreducible. Any irreducible unitary representation of G is, however, contained in U^{ξ} for some $\xi \in \hat{V}$ as an irreducible component. Let E be a function space on G. We define the Fourier transform T_f of $f \in E$ by $T_f(\xi) = \int_G f(g) U_g^{\xi} dg$. If f is integrable, this transform has meaning and T_f is a bounded operator valued function on $\hat{V}$.

The Plancherel formula for G (L_2-theory) was given by A. Kleppner and R. Lipsman ([1], Theorem 4.4). Let $C_c^{\infty}(G)$ be the space of all infinitely differentiable functions with compact support on G. Let $\mathcal{S}(G)$ be the space of all infinitely differentiable and rapidly decreasing functions on G. In this paper we consider these two cases $E = C_c^{\infty}(G)$ (the Paley-Wiener theorem) and $E = \mathcal{S}(G)$. Then $T_f(\xi)$ is an integral operator on $L_2(K)$ for any $f \in E$ and $\xi \in \hat{V}$ and its kernel function is given by $\kappa_f(\xi; k_1, k_2) = \int_V f(k_1 x, k_1 k_2^{-1}) e^{i<\xi, x>} dx$, $(k_1, k_2 \in K)$. When K is the identity group, κ_f is the ordinary Fourier transform on Euclidean space V. We call κ_f the scalar Fourier transform of f. Let $\tilde{E}$ and $\hat{E}$ be the images of E by the scalar Fourier transform and Fourier transform, respectively. The characterization of $\tilde{E}$ can be accomplished by the ordinary arguments of the classical Fourier analysis. To study the mapping $\kappa_f \mapsto T_f$ from $\tilde{E}$ to $\hat{E}$ we use an auxiliary theorem which can be proved using the representation theory of compact groups.

We can assume that there exists a K-invariant inner product on V. There-