J. Math. Soc. Japan
Vol. 27, No. 4, 1975

A characterization of arithmetic Fuchsian groups

By Kisao TAKEUCHI

(Received Feb. 28, 1975)

§1. Introduction.

Let k be a totally real algebraic number field of degree n. Then k has n distinct embeddings φ_i $(1 \leqq i \leqq n)$ into the real number field $\boldsymbol{R}$, where φ_1 is the identity. Let A be a quaternion algebra over k which is unramified at the place φ_1 and ramified at all other infinite places φ_i $(2 \leqq i \leqq n)$. Then there exists an $\boldsymbol{R}$-isomorphism

$$\rho: A \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{R} \longmapsto M_2(\boldsymbol{R}) \oplus \boldsymbol{H} \oplus \cdots \oplus \boldsymbol{H}, \tag{1}$$

where $\boldsymbol{H}$ is the Hamilton quaternion algebra.

Denote by ρ_i the composite of $\rho|_A$ with the projection to the i-th factor. Then ρ_1 (resp. ρ_i $(2 \leqq i \leqq n)$) is an isomorphism of A into $M_2(\boldsymbol{R})$ (resp. $\boldsymbol{H}$). By changing the indices suitably, for any element a of k we have

$$\rho_1(a \cdot 1_A) = a \cdot 1_2, \qquad \rho_i(a \cdot 1_A) = \varphi_i(a) \cdot 1_H \quad (2 \leqq i \leqq n), \tag{2}$$

where 1_A, 1_H and 1_2 are the unities of A, $\boldsymbol{H}$ and $M_2(\boldsymbol{R})$ respectively.

Denote by $\mathrm{tr}_A(\ )$ and $n_A(\ )$ (resp. $\mathrm{tr}_H(\ )$ and $n_H(\ )$) the reduced trace and the reduced norm of A (resp. $\boldsymbol{H}$). Then for any $\alpha \in A$, we have

$$\mathrm{tr}_A(\alpha) = \mathrm{tr}(\rho_1(\alpha)), \qquad \varphi_i(\mathrm{tr}_A(\alpha)) = \mathrm{tr}_H(\rho_i(\alpha)) \quad (2 \leqq i \leqq n), \tag{3}$$

$$n_A(\alpha) = \det(\rho_1(\alpha)), \qquad \varphi_i(n_A(\alpha)) = n_H(\rho_i(\alpha)) \quad (2 \leqq i \leqq n), \tag{4}$$

where $\mathrm{tr}(\ )$ and $\det(\ )$ are the trace and the determinant of $M_2(\boldsymbol{R})$ respectively.

Now take an order O of A and put

$$U = \{\varepsilon \in O \mid \varepsilon O = O \text{ and } n_A(\varepsilon) = 1\}.$$

Then U is a group called the unit group of O of norm 1. Denote by $\Gamma(A, O)$ the image $\rho_1(U)$ of U under ρ_1. Then $\Gamma(A, O)$ is a discrete subgroup of $SL_2(\boldsymbol{R})$. The group $SL_2(\boldsymbol{R})$ operates on the upper half plane $H = \{z \in \boldsymbol{C} \mid \mathrm{Im}(z) > 0\}$ in the following way:

$$SL_2(\boldsymbol{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$