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On a tensor product C^*-algebra associated with the free group on two generators

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Let G be the free group on two generators, and L^2 the Hilbert space of square summable complex valued functions on G. Let $\mathcal{L}$ and $\mathcal{R}$ be the C^*-algebras generated respectively by the left and right regular representations of G on L^2 and let $\mathfrak{A}$ be the C^*-algebra generated by $\mathcal{L}$ and $\mathcal{R}$ jointly. In [**1**] the authors provided a formula for computing the norm of certain operators in $\mathcal{L}$. In this paper the results of [**1**] are applied to the study of $\mathfrak{A}$, which may be regarded as a C^*-tensor product. (See the remark preceding Lemma 4.) We prove that $\mathfrak{A}$ contains the compact operators $\mathcal{C}$ in L^2 (Theorem 1) as its only closed two-sided ideal (Theorem 3), and that there is a derivation of $\mathfrak{A}$ into $\mathcal{C}$ which is not inner (Example 5). This investigation was suggested by Jun Tomiyama and Masamichi Takesaki at the Japan-U. S. Seminar on C^*-Algebras and Applications to Physics in Kyoto in May of 1974. Some related papers are listed in the references.

§1. Notation and Terminology.

Let S be a non-empty set. By $L^2(S)$ we mean the vector space of square summable complex valued functions on S. We prefer, however, to write the elements of $L^2(S)$ as (generally) infinite linear combinations, identifying the complex valued function f on S with the vector $\sum_{w\in S} f(w)w$. Thus we have

$$L^2(S)=\{\sum_{w\in S}\lambda_w w \mid \sum_{w\in S}|\lambda_w|^2<\infty\}\,.$$

$L^2(S)$ is a Hilbert space with inner product

$$(\sum_{w\in S}\lambda_w w,\ \sum_{w\in S}\mu_w w)=\sum_{w\in S}\lambda_w\bar{\mu}_w\,,$$

and resulting l_2 norm

$$\|\sum_{w\in S}\lambda_w w\|_2=(\sum_{w\in S}|\lambda_w|^2)^{\frac{1}{2}}.$$

By $L(S)$ we mean the subspace of $L^2(S)$ spanned by S; i.e., $L(S)$ consists of

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