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Lie algebra of vector fields and complex structure

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It was shown by [1] (also by [2] in compact case) that the structure of a smooth manifold M with countable basis is completely determined by the algebraic structure of the Lie algebra of smooth vector fields on M. In connection with this, K. Shiga posed the problem: whether or not the complex structure of a complex manifold is determined by the structure of the Lie algebra of vector fields of type (1, 0). The present paper is to give the affirmative answer to the problem together with some generalization. In this paper, all manifolds are assumed to have countable bases.

Let M be a complex manifold and $z_i = x_i + \sqrt{-1}\, y_i$ $(i=1, 2, \cdots, n)$ complex analytic coordinate in a neighbourhood of a point p of M. Complexified tangent vector at p is said to be of type (1, 0) if it is a complex linear combination of

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \qquad (i=1, 2, \cdots, n).$$

The set of all the tangent vectors of type (1, 0) costitutes a complex subbundle of the complexified tangent bundle of M. Smooth sections of this subbundle are called vector fields of type (1, 0), the totality of which forms a subalgebra $\mathfrak{A}_\partial(M)$ of the Lie algebra $\mathfrak{A}(M)$ of complex valued vector fields on M.

Now our main result can be formulated as follows:

THEOREM 1. *Let M and M' be complex manifolds and φ a Lie algebraic isomorphism of $\mathfrak{A}_\partial(M)$ to $\mathfrak{A}_\partial(M')$. Then there exists a biholomorphic map σ of M onto M' such that φ is induced by σ, that is,*

$$\varphi = \sigma_* .$$

Let us consider a more general situation. Let M be a smooth manifold. We denote by $C^\infty(M)$ the set of all real valued smooth functions on M. A real subalgebra A of $\mathfrak{A}(M)$ is said to be a quasi-foliation of M, if A satisfies the following conditions:

i) A is a module over $C^\infty(M)$, i.e., $X \in A$ implies $fX \in A$ for every $f \in C^\infty(M)$.

ii) For any point p of M, there exists $X \in A$ with $X_p \neq 0$.

iii) If $X_i \in A$ for $i=1, 2, \cdots$ and their supports forms a locally finite family,