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On the genus field of an algebraic number field of odd prime degree

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Let K be an algebraic number field of finite degree. Then the genus field $\tilde{K}$ of K is defined as the maximal abelian extension of K, which is a composite of an abelian extension $\tilde{k}_0$ of $\boldsymbol{Q}$ with K and is unramified at all the finite prime ideals of K (cf. Fröhlich [1]). The extension degree of $\tilde{K}$ over K is also called the genus number of K.

In the preceding paper [3], we have shown how we can construct explicitly the genus field $\tilde{K}$ of K, under the assumption that the degree and the discriminant of K are coprime.

The purpose of this paper is to determine the genus field and the genus number of an (arbitrary) algebraic number field K of odd prime degree l.

1. Let l be an odd prime number and let K be an algebraic number field of degree l.

Consider the p^n-th cyclotomic number field $k=\boldsymbol{Q}(\zeta_{p^n})$, where p is a prime number and ζ_{p^n} is a primitive p^n-th root of unity. Suppose that the decomposition of p in K as follows:

$$p=\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_m^{e_m}, \qquad N\mathfrak{p}_i=p^{f_i}, \tag{1}$$

where we have

$$\sum_{i=1}^{m} e_i f_i=[K:\boldsymbol{Q}]=l. \tag{2}$$

For a subfield k_0, of degree $d>1$, of $k=\boldsymbol{Q}(\zeta_{p^n})$, if the composite field k_0K is unramified (at all the finite prime ideals of K, i.e. at $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_m$), then, in (1), d divides $e_1, e_2, \cdots, e_m$ and so, by (2), d divides l i.e. we have $d=l$. So $m=1$, $e_1=l$, $f_1=1$, i.e. p is totally ramified in K. On the other hand, as d divides $\varphi(p^n)=p^{n-1}(p-1)=[k:\boldsymbol{Q}]$, there are two cases:

(i) $p\neq l$. Then $d=l$ divides $p-1$ and so we have $k_0\subset\boldsymbol{Q}(\zeta_p)$, i.e. k_0 is the unique subfield, of degree l, of $\boldsymbol{Q}(\zeta_p)$. In this case, as is shown in [3], the converse assertion holds. That is, if $p\equiv 1 \pmod{l}$ is totally ramified in K then k_0K is unramified over K.

(ii) $p=l$. Then we have $k_0\subset\boldsymbol{Q}(\zeta_{l^2})$, i.e. k_0 is the unique subfield, of de-