

On the riemannian structure all of whose geodesics are closed and of the same length

By Takashi SAKAI

(Received Nov. 26, 1973)

§ 1. Introduction.

Let (M, h_0) be an n -dimensional riemannian manifold all of whose geodesics are closed and of the same length $2\pi L$. We call such a riemannian structure a C_L -structure. In the present paper we shall give a characterization of C_L -structure (M, h_0) among all the riemannian structures on M .

Let $\text{Geod}(M, h_0)$ be the set of all oriented closed geodesics of (M, h_0) . Then $\text{Geod}(M, h_0)$ has a structure of compact $2(n-1)$ -dimensional differentiable manifold. Moreover on $\text{Geod}(M, h_0)$ there is the natural symplectic form Ω by which we may define the volume element

$$\omega := \{(-1)^{n(n-1)/2}/(n-1)!\} \Omega^{n-1}$$

on $\text{Geod}(M, h_0)$ (see § 2). $\mathcal{C}\mathcal{V}$ will denote the volume of $\text{Geod}(M, h_0)$ with respect to ω . Let \mathfrak{M}_M be the space of all riemannian structures g on M . Now we shall define the function f over \mathfrak{M}_M as follows:

$$f(g) := \text{vol}(M, g) / [c(M, g)]^n,$$

where $\text{vol}(M, g)$ is the volume of M with respect to the canonical measure ν_g derived from g and $c(M, g)$ is an average of the length of $c \in \text{Geod}(M, h_0)$ with respect to g , that is,

$$c(M, g) := (1/\mathcal{C}\mathcal{V}) \int_{\text{Geod}(M, h_0)} \left\{ \int_0^{2\pi L} \|\dot{c}(s)\|_g ds \right\} \omega.$$

In the above definition all geodesics $c \in \text{Geod}(M, h_0)$ are parametrized by the arc length relative to h_0 , and $\|\dot{c}(s)\|_g$ denotes the norm of the velocity vector $\dot{c}(s)$ of $c(s)$ with respect to g . Then f is a "smooth" function on \mathfrak{M}_M , i. e., for any differentiable one parameter family $g(t)$ of riemannian structures on M , $f(g(t))$ depends differentiably on t . Indeed, a critical point of f is a riemannian structure g on M such that $d/dt(f(g(t)))|_{t=0} = 0$ does hold for every differentiable one parameter family $g(t)$ with $g(0) = g$.

In the present paper we shall give a characterization of C_L -structure in terms of a critical point of the function f .

MAIN THEOREM. *Let (M, h_0) be a riemannian manifold with C_L -structure*