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Curvature and metric in Riemannian 3-manifolds

By Toshio NASU

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§ 1. Introduction.

Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian n-manifolds $(n \geqq 3)$ and f a diffeomorphism of (M, g) to $(\bar{M}, \bar{g})$. f is called a *curvature-preserving* diffeomorphism if for every point $p \in M$ and for every 2-plane section σ of the tangent space $T_p(M)$

$$\bar{K}(f_*\sigma) = K(\sigma)$$

holds, where K and $\bar{K}$ denote the sectional curvatures of (M, g) and $(\bar{M}, \bar{g})$, respectively. A point $p \in M$ is said to be *isotropic* if $K(\sigma) = \text{const.}$ for every 2-plane section σ of $T_p(M)$, and is said to be *non-isotropic* otherwise.

Recently, R. S. Kulkarni considered in [3] the converse of the *theorema egregium* of Gauss, which asserts that the curvature is a metric invariant, and proved that the curvature, in general, determines a conformal class of metric, that is, a curvature-preserving diffeomorphism $f: (M, g) \to (\bar{M}, \bar{g})$ is conformal if the set of non-isotropic points is dense in M (cf. Theorem 1 in [3]). It is natural to ask furthermore whether f is isometric or not. He showed in [3] that the answer to this question is affirmative if $n \geqq 4$ (cf. Fundamental Theorem in [3]), but he obtained only partial results for 3-manifolds assuming compactness and restricting sign of curvature (cf. § 6 in [3]). The purpose of this note is to give some affirmative answers to the above question for 3-manifolds.

In § 2 we shall prepare some general formulas on the conformal change of metric. In § 3, starting with Kulkarni's results, we shall obtain several lemmas on the curvature-preserving diffeomorphism f for later use. In § 4, after constructing a useful *constant associated with* f whose vanishing gives a necessary and sufficient condition for f to be isometric (cf. Theorem 1), we shall show as a corollary to Theorem 1 that the answer to the above question is also affirmative for conformally flat or compact 3-manifolds (cf. Corollary 1 and Corollary 2). Furthermore, as an application of Theorem 2 we shall give a partial result for complete manifolds with non-vanishing scalar curvatures (cf. Theorem 3). The hypothesis $n = 3$ is essential in § 4.

We shall assume, throughout this paper, that Riemannian manifolds under consideration are connected and of dimension $n \geqq 3$, their metrics are positive