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Pseudomonotone operators and nonlinear elliptic boundary value problems

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Introduction.

In this paper, from a view-point of the nonlinear operator theory we study nonlinear elliptic partial differential equations of the form

$$\text{(P)} \qquad -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} A_k(x, u, \nabla u) + A_0(x, u, \nabla u) = f \quad \text{in } \Omega$$

with some boundary conditions, and nonlinear elliptic variational inequalities of the form

$$\text{(V)} \qquad \begin{cases} u \in K, \\ a(u, u-v) - \int_\Omega f(u-v)dx \leqq \Phi(v) - \Phi(u) \quad \text{for all } v \in K, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary Γ, $f \in L^{p'}(\Omega)$ $(1/p+1/p'=1,\ 1<p<\infty)$, K is a convex closed subset of the Sobolev space $W^{1,p}(\Omega)$, Φ is a lower semicontinuous convex function on K and $a(\cdot, \cdot)$ is the functional on $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ given by

$$a(v, w) = \sum_{k=1}^{N} \int_\Omega A_k(x, v, \nabla v) \frac{\partial w}{\partial x_k} dx + \int_\Omega A_0(x, v, \nabla v) w dx .$$

In order to find a solution of (P) with a boundary condition of the Dirichlet type:

$$u|_\Gamma = \psi \quad \text{on } \Gamma,$$

one considers a variational inequality of type (V) with Φ and K associated with this boundary condition. Existence theorems for variational inequalities of type (V) were established by many authors (e.g., [1], [3], [4], [5], [6], [12], [14], [17], [21], [22]).

We treat partial differential equations of the form (P) with boundary conditions of mixed type. In particular, in case (P) has a smooth solution u, our boundary condition of mixed type is given by