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Pseudo-differential operators of multiple symbol and the Calderón-Vaillancourt theorem

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Introduction.

In the present paper we shall define a class $S_{\lambda,\rho,\delta}^{\tilde{m}_\nu;\tilde{m}'_\nu}$ of multiple symbol as an extension of the class $S_{\rho,\delta}^{m,m'}$ of double symbol in our previous paper [**6**], where $\tilde{m}_\nu=(m_1, \cdots, m_\nu)$ and $\tilde{m}'_\nu=(m'_0, m'_1, \cdots, m'_\nu)$ are real vectors and $m'_j \geqq 0$, $j=0, 1, \cdots, \nu$. The multiple symbol has the form $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)=p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)$ and the associated pseudo-differential operator $P=p(X^0, D_{\tilde{x}^\nu}, \tilde{X}^\nu)$ is defined as the map $P: \mathcal{B}\rightarrow\mathcal{B}$ by using oscillatory integrals developed in Kumano-go [**7**] and Kumano-go-Taniguchi [**8**], where $\mathcal{B}$ denotes the set of C^∞-functions with bounded derivatives of any order in R^n. Then, the (single) symbol $\sigma(P)(x, \xi)$ is given by $\sigma(P)(x, \xi)=e^{-ix\cdot\xi}P(e^{ix\cdot\xi})$.

We shall give a theorem which represents $\sigma(P)(x, \xi)$ by the oscillatory integral of the multiple symbol $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)$ and the asymptotic expansion formula for $\sigma(P)(x, \xi)$ will be given. As an application we shall prove the Calderón-Vaillancourt theorem in [**3**] (see also [**11**]) for the L^2-continuity of pseudo-differential operators of class $S_{\lambda,\delta,\delta}^0$ $(0\leqq\delta<1)$ only by symbol calculus. Another application is found in Tsutsumi [**10**], where our theorem is used to construct the fundamental solution $U(t)$ for a degenerate parabolic pseudo-differential operator in the class $S_{\rho,\delta}^0$ with a parameter t.

We believe that our theorem will be useful when we try to solve operator-valued integral equations with pseudo-differential operators as their kernels.

§1. Oscillatory integrals.

DEFINITION 1.1. We say that a C^∞-function $p(\eta, y)$ in $R_{\eta,y}^{2n}$ belongs to a class $\mathcal{A}_{\delta,\tilde{\tau}}^m$ for $-\infty<m<\infty$, $\delta<1$ and a sequence $\tilde{\tau}$; $0\leqq\tau_1\leqq\tau_2\leqq\cdots\leqq\tau_l\leqq\cdots$, when for any multi-index α, β we have

$$|p_{(\beta)}^{(\alpha)}(\eta, y)|\leqq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^{\tau_{|\beta|}} \tag{1.1}$$

for a constant $C_{\alpha,\beta}$ and set $\mathcal{A}_\delta=\bigcup_{-\infty<m<\infty}\bigcup_{\tilde{\tau}}\mathcal{A}_{\delta,\tilde{\tau}}^m$, where $p_{(\beta)}^{(\alpha)}=\partial_\eta^\alpha D_y^\beta p$, $D_{y_j}=-i\partial/\partial y_j$, $\partial_{\eta_j}=\partial/\partial\eta_j$, $j=1, \cdots, n$, $\langle y\rangle=\sqrt{1+|y|^2}$, $\langle\eta\rangle=\sqrt{1+|\eta|^2}$ (cf. [7], [8]).