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On the characterization of complex projective space by differential equations

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§ 1. Introduction.

The existence of a non-trivial solution of certain differential equations on a Riemannian manifold often determines some geometric and topological properties of the manifold. For example in [5] M. Obata announced the following results.

THEOREM A (see also [4]). *Let M^n be a complete connected Riemannian manifold of dimension $n \geqq 2$. Then M^n admits a non-trivial solution f of*

$$\nabla\nabla f + kfg = 0, \qquad k = \text{const.} > 0$$

if and only if M^n is globally isometric to a Euclidean sphere S^n of radius $1/\sqrt{k}$.

THEOREM B. *Let M^n be a complete connected, simply connected Riemannian manifold. Then M^n admits a non-trivial solution f of*

$$(\nabla\nabla\omega)(Z, X, Y) + k(2\omega(Z)g(X, Y) + \omega(X)g(Y, Z) + \omega(Y)g(X, Z)) = 0$$

where $\omega = df$ if and only if M^n is isometric to a Euclidean sphere of radius $1/\sqrt{k}$.

THEOREM C. *Let M^{2n} be a complete connected, simply connected Kähler manifold. Then M^{2n} admits a non-trivial solution f of*

$$4(\nabla\nabla\theta)(Z, X, Y) + c(2\theta(Z)G(X, Y) + \theta(X)G(Y, Z) + \theta(Y)G(X, Z)$$
$$- \theta(JX)\Omega(Y, Z) - \theta(JY)\Omega(X, Z)) = 0, \qquad c > 0$$

where $\theta = df$ if and only if M^{2n} is isometric to complex projective space PC^n with the Fubini-Study metric of constant holomorphic sectional curvature c.

In [5] Obata gives a proof of Theorem A and an indication of the proofs of Theorems B and C. Our purpose here is to show the relation between Theorems B and C by deducing Theorem C from Theorem B in the case of Hodge manifolds.

In Theorem B, grad f is an infinitesimal projective transformation and we show that on an odd-dimensional sphere S^{2n+1} we can find such a vector field orthogonal to the distinguished direction of the contact structure on