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Cohomology of Lie algebras over a manifold, I

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We shall here try to give a generalization of the de Rham theory towards higher order jet spaces. Our investigation is motivated by the recent works by M. V. Losik [12] and I. M. Gelfand and D. B. Fuks [7]. Their results, being entirely full of originality, might well be regarded as a part of a big theory to be formed on the theory of cohomology of vector fields which they have been rapidly developing since 1968. We shall, however, take somewhat different ways from theirs. Since the de Rham theory is of fundamental importance, we have more or less a prospect for obtaining a higher viewpoint in the manifold theory in our attempt to develop the de Rham theory. Actually, one of the most important contributions in this direction was made by Atiyah, Bott and Singer in the theory on elliptic complexes. Taking account of this successful theory, we are led to try a realization of a possibility of extending the de Rham theory from a point of view commanding both that of the cohomology of vector fields and that of the elliptic complexes. In a series of the papers, we shall give a constructive method of elliptic complexes in the jet spaces and investigate these complexes in various aspects.

We are going to describe our setting. Let M be a smooth manifold with a countable basis and E a smooth vector bundle over M. We denote by $\Gamma(E)$ the smooth cross-section space of E. We say that $\Gamma(E)$ is a Lie algebra over M if there is a Lie algebra structure on $\Gamma(E)$ endowed with the bracket rule $[\xi, \eta]$, satisfying a continuity condition and $\operatorname{supp}[\xi, \eta] \subset \operatorname{supp} \xi \cap \operatorname{supp} \eta$. Let W be another vector bundle over M. A Lie algebra representation φ of $\Gamma(E)$ to $\operatorname{Hom}(\Gamma(W), \Gamma(W))$ is called a differential representation if φ satisfies a continuity condition and $\operatorname{supp} \varphi(\xi)\eta \subset \operatorname{supp} \xi \cap \operatorname{supp} \eta$. By virtue of the cohomology theory of Lie algebras, we can then canonically construct a complex $\{C^p, d\}$ associated to the Lie algebra $\Gamma(E)$ and the representation φ. Here there is room for choice of the cochain spaces C^p. Indeed, it is natural to take as C^p the space of continuous alternating p-linear maps from $\Gamma(E) \times \cdots \times \Gamma(E)$ (p times) to $\Gamma(W)$. As far as we know, however, it seems to be as yet unsettled how to build up a reasonable cohomology theory corresponding to these cochain spaces.