J. Math. Soc. Japan
Vol. 26, No. 2, 1974

Ramanujan's formulas for L-functions

(To the memory of Professor Sigekatu Kuroda)

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(Received Aug. 24, 1972)
(Revised May 9, 1973)

Let $\zeta(s)$ be the Riemann's zeta-function. It is famous that $\zeta(2\nu)$, $0<\nu\in \boldsymbol{Z}$, is represented in terms of Bernoulli number and $\pi^{2\nu}$ and so is rational up to $\pi^{2\nu}$. But the numerical nature of $\zeta(2\nu+1)$, $\nu\geqq 1$, has long been unknown. As far as the author knows, only Ramanujan's formula*) is one involving $\zeta(2\nu+1)$.

Let χ be a non-principal primitive character mod k and $L(s, \chi)$ a Dirichlet L-function associated with χ. Then it is known that $L(2\nu, \chi)$, $\nu\geqq 1$, for even χ and $L(2\nu+1, \chi)$, $\nu\geqq 1$, for odd χ are represented by the generalized Bernoulli numbers in the sense of Leopoldt up to $\pi^{2\nu}$ and $\pi^{2\nu+1}$, respectively**). Analogously to the case of $\zeta(s)$, the numerical properties of $L(2\nu+1, \chi)$ for even χ and of $L(2\nu, \chi)$ for odd χ are unknown. Thus we are naturally led to ask "Ramanujan's formulas" for these values.

Now the purpose of the present paper is to formulate and prove "Ramanujan's formulas" for L-functions. Put

$$T_{\chi}=\sum_{h=0}^{k-1}\chi(h)e^{2\pi i h/k}.$$

Then for any $n>0$, we have

$$\chi(n)T_{\bar{\chi}}=\sum_{h=0}^{k-1}\bar{\chi}(h)e^{2\pi i n h/k}. \tag{0}$$

We define, for $0<a\in \boldsymbol{Z}$ and for $x>0$,

$$F_1(a, x, \chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^a}\frac{1}{e^{2\pi m x}-1}$$

and

$$F_2(a, x, \chi)=\sum_{h=0}^{k-1}\bar{\chi}(h)\sum_{n=1}^{\infty}\frac{1}{n^a}\frac{e^{2\pi n x h/k}}{e^{2\pi n x}-1}.$$

Then our formulas are formulated as follows:

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*) See for example [2].

**) The value $L(1, \chi)$ with odd or even χ is given in finite type at p. 336 of Borevich and Shafarevich's book "Number Theory, Academic Press, (1966)".