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On a bound for periods of solutions of a certain nonlinear differential equation (I)

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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§ 0. Introduction.

As is shown in [6], the nonlinear differential equation

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0\,,$$

where n is an integer $\geqq 2$, is the equation for the support function $x(t)$ of a geodesic in the 2-dimensional Riemannian manifold $\dot{O}_n^2$ with the metric:

$$\text{(0.1)} \qquad ds^2=(1-u^2-v^2)^{n-2}\{(1-v^2)du^2+2uv\,du\,dv+(1-u^2)dv^2\}$$

in the unit disk: $u^2+v^2<1$. Another geometric meaning of (E) is given in [4]. Any non constant solution $x(t)$ of (E) such that

$$x^2+\left(\frac{dx}{dt}\right)^2<1$$

is periodic and its period T is given by the improper integral:

$$\text{(0.2)} \qquad T=2\int_{a_0}^{a_1}\frac{dx}{\sqrt{1-x^2-C\left(\frac{1}{x^2}-1\right)^{\alpha}}}\,,$$

where

$$\text{(0.3)} \qquad C=(a_0^2)^{\alpha}(1-a_0^2)^{1-\alpha}=(a_1^2)^{\alpha}(1-a_1^2)^{1-\alpha}$$
$$(0<a_0<\sqrt{\alpha}<a_1<1,\ \alpha=1/n)$$

is the integral constant of (E) and $0<C<A=\alpha^{\alpha}(1-\alpha)^{1-\alpha}$.

Regarding T as a function of C, the following is known in [4]:

(i) T is differentiable and $T>\pi$,

(ii) $\lim_{C\to 0} T=\pi$ and $\lim_{C\to A} T=\sqrt{2}\,\pi$.

By means of a numerical analysis and observation about (E) in [5] and [7], M. Urabe conjectures the inequality