J. Math. Soc. Japan
Vol. 26, No. 1, 1974

Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$

By Hiroaki HIJIKATA

(Received July 8, 1972)
(Revised June 18, 1973)

0.0. Let $\Gamma=\Gamma_0(N)$ be the congruence subgroup of level N, i.e. the group consisting of all two by two integral unimodular matrices $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c\equiv 0 \bmod N$. χ being a multiplicative character modulo N, let $S_0(\Gamma, \{k\}, \chi)$ be the space of cusp forms of weight k, and let $T(n)$ be the Hecke operator acting on $S_0(\Gamma, \{k\}, \chi)$ (see 1.1 and 5.5 for the definition).

When N is square free, M. Eichler ([3], [4]) gave an explicit formula for the trace $\operatorname{tr} T(n)$ of $T(n)$ on $S_0(\Gamma, \{k\}, \chi)$ with several interesting applications, and he suggested ([3] p. 168–169) that it might be interesting to refine the arithmetic of quaternion algebras so that one can handle the square level cases or principal congruence subgroups. Since then several authors took up the problem. For example, H. Shimizu [9] generalized Γ to be the higher dimensional (Hilbert modular type) ones (still with square free level), M. Yamauchi [11] generalized Γ to be the ones with level $4N'$ (where N' is odd and square free).

In this paper, we have again taken up the problem suggested by Eichler. We can give an explicit formula of $\operatorname{tr} T(n)$ for $\Gamma_0(N)$ (and its normal subgroups of Fricke type) for arbitrary N and for their analogues obtained from indefinite quaternions. In the following, we shall write down a 'ready to compute' formula of $\operatorname{tr} T(n)$ for $\Gamma_0(N)$ and its normal subgroups $\Gamma(\mathfrak{h})$ defined as follows. Let M be a divisor of N, let $\mathfrak{h}$ be a subgroup of $(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}$, and let $\Gamma=\Gamma(\mathfrak{h})$ denote the subgroup of $\Gamma_0(N)$ consisting of the elements $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a lies in $\mathfrak{h}$ modulo M. (For example, if we take $N=M^2$ and $\mathfrak{h}=\{1\}$, then $\Gamma(\mathfrak{h})$ is conjugate to the principal congruence subgroup $\Gamma(M)$.) Let χ be a character mod M, also considered as a linear character of $\Gamma(\mathfrak{h})$ via $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto\chi(a)$. Assume n is prime to the level N and $k\geqq 2$, then, for any (M and) $\mathfrak{h}$, the trace of $T(n)$ acting on $S_0(\Gamma(\mathfrak{h}), k, \chi)$ is given by the following theorem.