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On the Euler integral representations of hypergeometric functions in several variables

By Akio HATTORI and Tosihusa KIMURA

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§1. Statement of the problem.

It is well known that the hypergeometric function $F(\alpha, \beta, \gamma, x)$ defined by the series

$$F(\alpha, \beta, \gamma, x)=\sum_{m=0}^{\infty}\frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)}x^m \tag{1.1}$$

has the Euler integral representation

$$F(\alpha, \beta, \gamma, x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}dz, \tag{1.2}$$

where (a, k) denotes the factorial function

$$a(a+1)\cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}$$

Series (1.1) is meaningful provided that $\gamma\neq 0, -1, -2, \cdots$, and then the radius of convergence is one except in the case when either α or β is a non positive integer. On the other hand, integral (1.2) is convergent if

$$0<\mathrm{Re}\,\beta<\mathrm{Re}\,\gamma, \tag{1.3}$$

and then integral (1.2) is holomorphic with respect to x in $\boldsymbol{C}-[1, \infty)$, $\boldsymbol{C}$ being the set of complex numbers.

It is natural to attempt weakening restriction (1.3) in the integral representation. Indeed, two methods for that are known: method of double contour integrals and that of the finite part for divergent integrals. The former is usually carried out as follows. Suppose that $x\in\boldsymbol{C}-[1, \infty)$. Let a be a point in $\boldsymbol{C}-\{0, 1, 1/x\}$, say lying on the real axis between 0 and 1, and let l_0 and l_1 be loops in the positive direction at a in $\boldsymbol{C}-\{0, 1, 1/x\}$, l_0 encircling only $z=0$ and l_1 encircling only $z=1$. Form the contour C consisting of l_0, l_1, l_0^{-1} and l_1^{-1} in this order:

$$C=l_0l_1l_0^{-1}l_1^{-1},$$

l_0^{-1} and l_1^{-1} being the inverse loops of l_0 and l_1 respectively. Then take a