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The nilpotency of elements of the stable homotopy groups of spheres

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§ 0. Introduction.

In this paper we shall describe the theory of extended powers of CW-complexes. Our main application is to demonstrate the following conjecture of M. G. Barratt:

CONJECTURE. *Any element of positive stem of the stable homotopy groups of spheres is nilpotent.*

The extended n-th power $D_n(X)$ of a CW-complex X is defined by $WS_n \times_{S_n} X^{(n)} / WS_n \times_{S_n} (\textit{base point})^{(n)}$, where WS_n is an acyclic S_n-free complex as S_n being the n-th symmetric group (for details of the definition, see § 1). The study of constructions of this kind was initiated by N. E. Steenrod [18]. For $n =$ a prime, various applications of extended powers to homotopy theory have been done by J. F. Adams, M. G. Barratt, D. S. Kahn, M. Mahowald and H. Toda. Also R. J. Milgram treated the D_4 construction to apply it to the Arf invariant problem.

The basic idea of the proof of the conjecture is given by H. Toda in [19]. That is, roughly speaking, the study of the stable homotopy type of $D_n(X)$ for $X = S^k$ or $S^k \cup_p e^{k+1}$ may lead us to the conjecture. We shall describe two ways of attacking the conjecture, which are given in Part I and Part II, respectively. The second method gives the comprehensive result (Corollaries 8.2 and 8.4), but the estimate of exponent t in $\alpha^t = 0$ is very large. On the other hand, though it gives only a restricted result (Corollary 4.2), the first method gives much better estimate of exponent than that of the second one.

The paper is organized as follows.

§ 1. Extended power of CW-complexes.
§ 2. Cohomology group of $D_n(X)$.
§ 3. Homotopy type of Γ-spectrum.
§ 4. Applications to the stable homotopy groups of spheres.
§ 5. Further properties of $D_n(X)$.
§ 6. Some properties of h_p.