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On the factors of the jacobian variety of a modular function field

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§ 0. Introduction.

For a positive integer N, put

$$\Gamma_0(N)=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\boldsymbol{Z})\mid c\equiv 0 \pmod{N}\right\},$$

$$\Gamma_1(N)=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \Gamma_0(N)\mid a\equiv d\equiv 1 \pmod{N}\right\}.$$

We consider any group Γ such that $\Gamma_1(N)\subset\Gamma\subset\Gamma_0(N)$, and call it *a group of level N*. Let J denote the jacobian variety of the compact Riemann surface $\mathfrak{H}^*/\Gamma$, where $\mathfrak{H}^*$ means the union of the upper half plane

$$\mathfrak{H}=\{z\in \boldsymbol{C}\mid \mathrm{Im}(z)>0\}$$

and the cusps of Γ. Further let $S_k(\Gamma)$ be the vector space of all holomorphic cusp forms of weight k with respect to Γ. Then, with each common eigenfunction $f(z)$ in $S_2(\Gamma)$ of the Hecke operators T_n for all n, one can associate an abelian variety A that is a "factor" of J. The purpose of this note is to consider a few arithmetical questions concerning the correspondence between f and A. Besides, as an application of our methods, we shall give a proof of Dirichlet's class number formula for an imaginary quadratic field, without using the residue technique.

We start our treatment by proving that A can naturally be obtained as a *quotient* of J by an abelian subvariety rational over $\boldsymbol{Q}$ (Theorem 1). Actually in [**11**, § 7.5], we gave a formulation with such a factor as a *subvariety* of J. The two formulations are essentially equivalent, but there is a subtle difference. At any rate, they are connected by the following fact: there is a canonical $\boldsymbol{C}$-linear isomorphism of $S_2(\Gamma)$ onto the tangent space of J at the origin, which has a certain commutative property with the action of Hecke operators. Such an isomorphism was used in the proofs of [**11**, Th. 7.14, Prop. 7.19] and also in [**12**], but not explicitly given. This point will be clarified in § 2. It will be shown in § 3 that A can be obtained as a complex torus whose periods are those of f and some other cusp forms. We shall consider