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On the representations of an integer as a sum of two squares and a product of four factors

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§ 1. Introduction.

1.1. The purpose of the present paper is to establish the asymptotic formula for the number of representations of an integer as a sum of two integral squares and a product of four positive integral factors.

Our problem is obviously equivalent to the study of the asymptotical behaviour of the sum

$$\sum_{n<N} r(N-n)d_4(n) \qquad (\text{as } N\to\infty), \tag{1}$$

where $r(n)$ and $d_4(n)$ stand for the number of representations of n as a sum of two squares and as a product of four factors, respectively.

Our problem and the so-called additive divisor problem are similar in that each sum can be expressed as a combination of sums of iterated divisor functions over arithmetic progressions with variable modulus, whose size depends on the parameter N. But our problem has much greater difficulty caused mainly by the inner structure of $r(n)$. The same fact has been already noticed by Hooley [1] between the divisor problem of Titchmarsh and a conjecture of Hardy and Littlewood. Hence our proof depends on various devices of Hooley, and also the large sieve method plays an important role in this paper.

1.2. Notation: To avoid the unnecessary complications we assume that throughout this paper the parameter N is a sufficiently large *odd* integer.

ε is assumed to be positive and sufficiently small, and the constants in the symbols "O" and "$\ll$" depend on ε at most.

(m, n) stands for the greatest common divisor of m and n. A prime number is denoted by p, and $p^\alpha \| n$ means that p^α is the highest power of p which divides n. The symbol $m \subset n$ indicates that all prime divisors of m divide n.

$\omega(n)$ and $\Omega(n)$ are respectively the numbers of different prime factors of n and the total number of prime factors of n. $d(n)$ is the number of divisors of n, and $d_k(n)$ is the number of representations of n as a product of k factors.