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Quasi-permutation modules over finite groups

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Let Π be a finite group. A finitely generated Z-free Π-module is briefly called a Π-module. A Π-module is called a permutation Π-module if it is expressible as a direct sum of some $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence: $0\to M\to S\to S'\to 0$ where S and S' are permutation Π-modules.

In [**2**] we have studied the properties of quasi-permutation modules in relation with a problem in invariant theory. In this paper we will give some basic results on quasi-permutation modules as a continuation to [**2**].

First we will consider projective quasi-permutation Π-modules.

Let R be a Dedekind domain and K be the quotient field of R. Let Σ be a separable K-algebra and Λ be an R-order in Σ. Denote by $\boldsymbol{P}(\Lambda)$ the set of all isomorphism types of finitely generated projective (left) Λ-modules and put $\boldsymbol{P}_0(\Lambda)=\{[P]\in \boldsymbol{P}(\Lambda)\,|\,P \text{ is locally free}\}$. Let $P_0(\Lambda)$ be the Grothendieck group of $\boldsymbol{P}_0(\Lambda)$. We define an epimorphism $\mu_\Lambda\colon P_0(\Lambda)\to Z$ by $\mu_\Lambda([P_1]-[P_2])=\operatorname{rank}_\Sigma KP_1-\operatorname{rank}_\Sigma KP_2$. Now we put $C(\Lambda)=\operatorname{Ker}\mu_\Lambda$ and call this the (reduced) projective class group of Λ (cf. [**5**], [**11**]). Especially, if Λ is commutative, then $C(\Lambda)$ is isomorphic to the Picard group of Λ. Further let Ω be a maximal R-order in Σ which contains Λ. We define a homomorphism: $\nu_{\Omega/\Lambda}\colon C(\Lambda)\to C(\Omega)$ by $\nu_{\Omega/\Lambda}([P_1]-[P_2])=[\Omega\otimes_\Lambda P_1]-[\Omega\otimes_\Lambda P_2]$. Then it is known that $\nu_{\Omega/\Lambda}$ is an epimorphism but not always a monomorphism. Hence putting $\tilde{C}(\Lambda)=\operatorname{Ker}\nu_{\Omega/\Lambda}$, we have an exact sequence:

$$0\longrightarrow \tilde{C}(\Lambda)\longrightarrow C(\Lambda)\longrightarrow C(\Omega)\longrightarrow 0\,.$$

Especially let $\Lambda=Z\Pi$ and let Ω_Π be a maximal order in $Q\Pi$ which contains $Z\Pi$. Then, by the Swan's theorem ([**11**]), we have $\boldsymbol{P}_0(Z\Pi)=\boldsymbol{P}(Z\Pi)$ and $\tilde{C}(Z\Pi)=\{[\mathfrak{a}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{a}$ is a projective (left) ideal of $Z\Pi$ such that $\Omega_\Pi\mathfrak{a}\oplus\Omega_\Pi\cong\Omega_\Pi\oplus\Omega_\Pi$ as Ω_Π-modules$\}$. It is noted that $\tilde{C}(Z\Pi)$ does not depend on the choice of Ω_Π (cf. [**3**]). On the other hand, we put $C^q(Z\Pi)=\{[\mathfrak{a}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{a}$ is a quasi-permutation projective (left) ideal of $Z\Pi\}$. Then it is easily seen that $C^q(Z\Pi)$ is also a subgroup of $C(Z\Pi)$.

Let Π be a cyclic group of order n and σ be a generator of Π. We