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A generalised combinatorial distribution problem

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§ 1. Introduction and summary.

Let $A=(a_{ij})$ be a square matrix of size n and let the entries of A be non-negative integers. Denote the sum of row i of A by r_i, $r_i \geqq 0$, and that of the column j of A by s_j, $s_j \geqq 0$. If T denotes the total sum in A, then it is clear that

$$T=\sum_{i=1}^{n} r_i=\sum_{j=1}^{n} s_j\,. \tag{1.1}$$

We call $R=(r_1, r_2, \cdots, r_n)$ the row sum vector and $S=(s_1, s_2, \cdots, s_n)$ the column sum vector of A. The vectors R and S determine a class

$$G=G(R, S)\,, \tag{1.2}$$

consisting of all such matrices of size n, with row sum vector R and column sum vector S. For A admitting integers 0 and 1 only, known as (0, 1)-matrix, many diversified topics including traces, term ranks, widths, heights, and combinatorial designs related to problems dealing with a class G', a subclass of G, consisting of (0, 1)-matrices, have attracted the attention of many authors. Among them are Ryser (1957, 1960a, 1960b), Jurkat and Ryser (1967), and Murty (1968). A detailed list of references may be found in Ryser (1960a).

Let $H(n, R, S)$ denote the number of members of class G, that is the number of ways in which n distinct things, the j-th replicated s_j times, $s_j \geqq 0$, can be distributed among n persons, the i-th getting r_i, $r_i \geqq 0$. The case, when each row sum and column sum equals $r(\geqq 1)$, and the number $H(n, R, R)$ denoted by $H(n, r)$, has been investigated by Kenji Mano (1961). He gives an intricate formula for $r=2$. Anand et al. (1966) extended the result to $H(3, r)$ and stated a plausible formula for $H(n, r)$. Recently, Nath and Iyer (1972) have suggested the use of the generating functions to expedite calculations and obtained explicit formulae for $H(3, r)$ and $H(4, r)$.

In the present paper, we give some inequalities for $H(n, R, S)$, true for all positive n, and an explicit formula for $H(3, R, S)$. The procedure applies to rectangular matrices as well as square ones.