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Ergodic theorems and weak mixing for Markov processes

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§ 1. Definitions and notation.

A Markov process is defined to be a quadruple $(\Omega, \mathcal{B}, m, P)$ where $(\Omega, \mathcal{B}, m)$ is a σ-finite measure space with positive measure m and where P is a positive linear contraction on $L^1(\Omega)$. P will be written to the right of its variable, and the adjoint in $L^\infty(\Omega)$ will also be denoted by P but will be written to the left of its variable. Thus $\langle uP, f\rangle = \langle u, Pf\rangle$ for $u \in L^1(\Omega)$ and $f \in L^\infty(\Omega)$. A σ-finite positive measure λ on $(\Omega, \mathcal{B})$ absolutely continuous with respect to m is called *subinvariant* if $\int P1_A(\omega)\lambda(d\omega) \leqq \lambda(A)$ for any $A \in \mathcal{B}$ and *invariant* if $\int P1_A(\omega)\lambda(d\omega) = \lambda(A)$ for any $A \in \mathcal{B}$. Throughout this paper m is assumed to be either an infinite subinvariant measure or a finite invariant measure. It is well known that P on $L^\infty(\Omega)$ is also a linear contraction on $L^1(\Omega)$ and hence it may be considered to be a linear contraction on each $L^p(\Omega)$ with $1 \leqq p \leqq \infty$ by the Riesz convexity theorem. The adjoint process of $(\Omega, \mathcal{B}, m, P)$ will be denoted by $(\Omega, \mathcal{B}, m, P^*)$; its properties are studied in [4, Chapter VII].

The process $(\Omega, \mathcal{B}, m, P)$ is called

1) *ergodic*, if $P1_A = 1_A$ implies $m(A) = 0$ or $m(\Omega - A) = 0$;
2) *weakly mixing*, if

$$L^2(\Omega) \ominus \left\{ f \in L^2(\Omega);\ \lim_n \frac{1}{n} \sum_{i=0}^{n-1} |\langle P^i f, f\rangle| = 0 \right\}$$

is so small as to contain nothing more than the constant functions;

3) *strongly mixing*, if

$$L^2(\Omega) \ominus \{ f \in L^2(\Omega);\ \lim_n \langle P^n f, f\rangle = 0\}$$

is so small as to contain nothing more than the constant functions.

We note that our definition of strong mixing is due to Foguel [4] and coincides with the notion of "mixing" proposed by Lin [7].