J. Math. Soc. Japan
Vol. 25, No. 1, 1973

Rings satisfying polynomial constraints

By Mohan S. PUTCHA and Adil YAQUB

(Received Feb. 12, 1972)

§ 1. Introduction.

In a well-known paper [1] Herstein proved that if an associative ring R has the property that for each x in R there exists a polynomial $f_x(\lambda)$ (depending on x) with integer coefficients such that $x-x^2 f_x(x)$ is in the center of R, then R is commutative. In this paper, we generalize Herstein's Theorem by essentially considering conditions on n elements $x_1, \cdots, x_n$ of R. We make extensive use of Herstein's methods throughout. A related problem has been recently investigated by the authors [5].

§ 2. Main results.

Throughout, R is an associate ring and $x_1, \cdots, x_n$ are elements of R. A *word* $w(x_1, \cdots, x_n)$ is simply a product in which each factor is x_i, for some $i=1, \cdots, n$. A *polynomial* $f(x_1, \cdots, x_n)$ is, then, an expression of the form $f(x_1, \cdots, x_n)=c_1 w_1(x_1, \cdots, x_n)+\cdots+c_q w_q(x_1, \cdots, x_n)$, where the c_i are integers.

DEFINITION. Let n be a positive integer. An α_n-ring is an associative ring R with the property that for all $x_1, \cdots, x_n$ in R, there exists a polynomial $f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$ (depending on $x_1, \cdots, x_n$) with integer coefficients such that: (a) degree of each x_i in every term of $f_{x_1,\cdots,x_n}(x_1, \cdots, x_n) \geqq 2$, and (b) $x_1 \cdots x_n - f_{x_1,\cdots,x_n}(x_1, \cdots, x_n) \in Z$, where Z denotes the center of R.

It is clear that subrings and homomorphic images of α_n-rings are again α_n-rings.

Our present object is to prove the following

THEOREM (Principal Theorem). *If R is an α_n-ring with center Z, then $R^n \subseteqq Z$ (and conversely).*

Since this theorem is true for $n=1$ (Herstein's Theorem), we shall assume that $n>1$ and

(2.0) FUNDAMENTAL INDUCTION HYPOTHESIS. The above theorem is true for α_{n-1}-rings.

In preparation for the proof of this theorem, we first establish the following lemmas.