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Products of two semi-algebraic groups

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§ 1. Introduction.

Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the field of real numbers and complex numbers, respectively. For a field Φ, we denote by $GL(n, \Phi)$ the group of all n by n non-singular matrices over Φ. A subgroup A of $GL(n, \Phi)$ is called *algebraic* if there exists a family of polynomials of n^2 variables over Φ which defines A. In this paper, we are mainly interested in subgroups of $GL(n, \boldsymbol{R})$. A subgroup P of $GL(n, \boldsymbol{R})$ is said to be *pre-algebraic*[2)] if there exists an algebraic subgroup A of $GL(n, \boldsymbol{R})$ which contains P as a subgroup of finite index. A pre-algebraic group is closed, and a closed subgroup G of $GL(n, \boldsymbol{R})$ is pre-algebraic if and only if the Lie algebra $\mathfrak{G}$ of G is algebraic and G has only finitely many connected components. For any subgroup G of $GL(n, \boldsymbol{R})$, we can find the smallest pre-algebraic group $\mathcal{A}(G)$ containing G. Let us call $\mathcal{A}(G)$ the *pre-algebraic hull* of G. For a topological group G, we adopt the notation G_e for the identity component group of G. The identity component of $\mathcal{A}(G)$ will be denoted by $\mathcal{A}_e(G)$, i.e. $\mathcal{A}_e(G) = (\mathcal{A}(G))_e$.

Let G be a connected Lie subgroup of $GL(n, \boldsymbol{R})$. Then the pre-algebraic hull $\mathcal{A}(G)$ is connected, and the commutator subgroup of $\mathcal{A}(G)$ is closed and is contained in G. In Goto [4], the author defined G to be semi-algebraic if G contains a maximal compact subgroup of $\mathcal{A}(G)$. If G is semi-algebraic, then G is a closed normal subgroup of $\mathcal{A}(G)$ such that the factor group $\mathcal{A}(G)/G$ is isomorphic with a vector group $\boldsymbol{R}^k$ of a certain dimension k, and vice versa. Let us extend the definition of semi-algebraic groups to the non-connected case.

DEFINITION. A closed subgroup S of $GL(n, \boldsymbol{R})$ is said to be *semi-algebraic* if S is a normal subgroup of its pre-algebraic hull $\mathcal{A}(S)$ and the factor group $\mathcal{A}(S)/S$ is a vector group.

It is obvious that this generalizes the definition of the connected case and that a pre-algebraic group is semi-algebraic.

Let S be a semi-algebraic group. Then $S\mathcal{A}_e(S)$ is of finite index in $\mathcal{A}(S)$,

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2) About pre-algebraic groups, see Goto-Wang [5].