J. Math. Soc. Japan
Vol. 24, No. 3, 1972

Decomposition theorem for Hopf algebras and pro-affine algebraic groups

By Eiichi ABE and Yukio DOI

(Received July 26, 1971)
(Revised March 4, 1972)

§ 0. Introduction.

Generalizing the natural correspondence between affine algebraic groups over an algebraically closed field k and their coordinate rings, we have an anti-equivalent correspondence between the category of pro-affine algebraic groups over k and the category of commutative reduced Hopf algebras (i. e. which have no nilpotent elements other than 0) over k. (See [**1**], [**2**], [**3**].)

The purpose of the paper is to discuss relations between the properties of these two objects and especially to obtain certain properties of groups from those of their Hopf algebras.

In the first two sections, we reproduce some known relations between the properties of commutative reduced Hopf algebras and pro-affine (or affine) algebraic groups (cf. [**3**]), and in § 3 give a certain property of the co-radical of a Hopf algebra. Sections § 4 and § 5 are devoted to discuss general commutative Hopf algebras over an arbitrary field. In § 4, we give the definition of semi-direct product of Hopf algebras which is the dual of smash product in the sense of Sweedler [**3**]. In § 5, we give a decomposition theorem for Hopf algebras. In § 6, we give definitions and properties of exact sequences of reduced Hopf algebras which are dual of those of groups. When the sequence splits, we may apply to it the decomposition theorem given in § 5. It is well known that a connected affine algebraic group over an algebraically closed field of characteristic zero is the semi-direct product of the unipotent radical and a linearly reductive subgroup. Applying the decomposition theorem for Hopf algebras, the decomposition theorem for pro-affine (or affine) algebraic groups can be described in terms of Hopf-algebra theory.

Throughout the paper, all Hopf algebras are commutative over a field k.

§ 1. Preliminaries.

(1.1) Let V be a vector space over a fixed ground field k, and let $V^* = \mathrm{Hom}_k(V, k)$ be the linear dual space. For $f \in V^*$, $v \in V$, we will usually