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On smooth extension theorems

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0° Introduction.

It is well known that the space $\boldsymbol{F}$ of the smooth functions of a compact smooth manifold M is a Frechet space and leads a Sobolev chain $\{F^s\}$, namely, F^s is a Hilbert space obtained by the completion of $\boldsymbol{F}$ in a norm involving the integral of squares of all derivatives up to order s, and $\boldsymbol{F}$ is the inverse limit of the system $\{F^s\}$. Of course the addition $(a, b)\rightarrow a\pm b$ in $\boldsymbol{F}$ can be extended to the smooth mapping of $F^s\times F^s$ into F^s for every s.

Assume M is closed (that is, compact without boundary). Then the connected component $\mathcal{D}_0$ of the group of the diffeomorphisms of M in C^∞-topology has similar properties as above, that is, (1) $\mathcal{D}_0$ is a Frechet Lie group [**4**] and (2) there exists a system $\{\mathcal{D}_0^s\}$, $s\geqq \dim M+5$, of smooth Hilbert manifolds each of which is a topological group such that $\mathcal{D}_0$ is the inverse limit of the system $\{\mathcal{D}_0^s\}$ [**2**, **7**, **9**]. Though the group operations of $\mathcal{D}_0^s$ is the extension of that of $\mathcal{D}_0$, the differentiability of these is not so simple. For example (a) the multiplication $(g, h)\rightarrow gh$ of $\mathcal{D}_0$ can be extended to the C^l-map of $\mathcal{D}_0^{s+l}\times\mathcal{D}_0^s$ into $\mathcal{D}_0^s$. On the other hand, (b) the right translation $R_g:\mathcal{D}_0^s\rightarrow\mathcal{D}_0^s$ is smooth for any $g\in\mathcal{D}_0^s$ [**2**, **9**]. As a matter of fact, these properties can be proven in case that M has a boundary and $s\geqq\dim M+1$ (c. f. [**2**]).

In this paper as well as the previous paper [**9**], the author restricted his concern to the case that (a) M has no boundary, (b) $s\geqq\dim M+5$ and (c) the connected component $\mathcal{D}_0$ of the total group of the smooth diffeomorphisms. The reason is the following:

(a) If one constructs an abstract group theory having the properties mentioned above (and this is what he wants to do in the future), then the boundary cases will come in it very naturally.

(b) If $s\geqq\dim M+5$, then we have a nicer property with respect to the composition of maps. Actually, we have a useful inequality in this case (see Theorem A in [**9**]).

(c) If one concerns with only local properties, it is enough to treat a neighbourhood of the identity. So as a group generated by a neighborhood of the identity, we have only to consider the connected component $\mathcal{D}_0$.