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A note on the prime radical

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§ 1. Introduction.

In 1943 R. Baer introduced the lower nil radical, which is commonly called the prime radical, as a radical built from nilpotent rings [**1**]. N. McCoy first considered the intersection of prime ideals of a ring [**7**] and then J. Levitzki showed that the prime radical was the intersection of the prime ideals of a ring [**6**]. Elementwise characterizations of the prime radical were given first by N. Jacobson in terms of m-sequences [**4**, p. 195] and then recently by J. Lambek in terms of strongly nilpotent elements [**5**, p. 55]. This latter characterization enables us to describe it in terms of annihilators. The primary purpose of this paper is to give necessary and sufficient conditions in terms of annihilators for the prime radical to be nilpotent (see Corollary 3). These conditions follow from Theorem 2. This also proves that the prime radical of a ring with the minimum condition on (two-sided) ideals is nilpotent (see Corollary 5).

§ 2. The results.

R will always denote a ring and $r(S)$ ($l(S)$) the right (left) annihilator of a subset S of R. For $b \in R$ we write $r(b)$ instead of $r(\{b\})$. Also, bR means the right ideal generated by b.

A decreasing sequence of sets $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ of R is said to have a *right (left) constant annihilator* provided that the corresponding increasing sequence of right (left) annihilators becomes constant, that is, $r(S_n) = r(S_{n+j})$ for some fixed n and all $j \geqq 1$. ($l(Sn) = l(S_{n+j})$). In particular we say that a sequence of elements $\{x_i\}$ has a *right constant annihilator* if $Rx_1 \supseteq Rx_2x_1 \supseteq Rx_3x_2x_1 \supseteq \cdots$ has a right constant annihilator.

PROPOSITION 1. *The prime radical is the set S of elements x such that xR is nil and each sequence $\{x_i\}$ where $x_1 = x$, $x_{k+1} \in x_k \cdots x_1R$ has a right constant annihilator.*

PROOF. All strongly nilpotent elements belong to S. Let $y \in S$ and let $\{y_i\}$ be a sequence where $y_1 = y$, $y_{k+1} = y_k \cdots y_1 p_k$ for some $p_k \in R$ and for all