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On zeta-theta functions

(To the memory of Professor M. Sugawara)

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§ 0. Introduction.

In our previous paper [1], we treated certain zeta-functions attached to symmetric tensor representations of odd degrees of the group $G=SL(2, \boldsymbol{R})$. In the present paper, we deal with analogous functions connected with symmetric tensor representations of *even* degrees of the same group G.

Let M_ν^* be the "modified" symmetric tensor representation of even degree $\nu \geqq 2$ of G (Cf., 1.3). Then $M_\nu^*(\sigma)$, $\sigma \in G$, leaves an indefinite symmetric matrix S_ν invariant and so $M_\nu^*(G)$ is contained in the orthogonal group $\tilde{G}_\nu$ of S_ν. Let K be the orthogonal subgroup of G (which is a maximal compact subgroup of G) and $\tilde{K}_\nu$ be a maximal compact subgroup of $\tilde{G}_\nu$ containing $M_\nu^*(K)$. To determine $\tilde{K}_\nu$, we take and fix a definite symmetric matrix P_ν which is a "majorant" for S_ν (Cf., 1.2). Now $\tilde{H}_\nu=\tilde{K}_\nu\backslash\tilde{G}_\nu$ has a structure of Riemannian symmetric space, called the representation space of $\tilde{G}_\nu$ by Siegel [3]. Let $H=\{z \in \boldsymbol{C} \mid \operatorname{Im} z>0\}$ be the usual upper half plane. Then $H=K\backslash G$. Using P_ν and M_ν^*, we can define an imbedding φ_ν of H into $\tilde{H}_\nu$ (Cf., 1.3).

Let $\tilde{f}_\mathfrak{a}(\omega, \mathfrak{H})$ be the Siegel's theta-function defined on $H\times\tilde{H}_\nu \ni (\omega, \mathfrak{H})$, attached to our indefinite S_ν, where $\mathfrak{a}$ is a rational vector such that $2S_\nu\mathfrak{a}$ is integral (Cf., 2.1 or 2.3). Let $\mathfrak{a}_0, \cdots, \mathfrak{a}_t$ be a complete set of representatives mod 1 of rational vectors $\mathfrak{a}$ with integral $2S_\nu\mathfrak{a}$ and $\tilde{f}(\omega, \mathfrak{H})$ be the vector with components $\tilde{f}_{\mathfrak{a}_0}, \cdots, \tilde{f}_{\mathfrak{a}_t}$. We denote with $f_\mathfrak{a}(\nu; \omega, z)$ and $f(\nu; \omega, z)$ the pull-back of $\tilde{f}_\mathfrak{a}(\omega, \mathfrak{H})$ and $\tilde{f}(\omega, \mathfrak{H})$ to H by φ_ν, respectively. Then $f_\mathfrak{a}(\nu; \omega, z)$ is a non-holomorphic function defined on $H\times H \ni (\omega, z)$. Let $\mathfrak{F}_\mathfrak{a}(\nu)$ be a fundamental domain on H for the group $\Gamma_\mathfrak{a}(\nu)=\{\sigma \in SL(2, \boldsymbol{Z}) \mid M_\nu^*(\sigma)\mathfrak{a} \equiv \mathfrak{a} \pmod 1\}$. Then $f_\mathfrak{a}$, as a function of the second argument z, is invariant by $\Gamma_\mathfrak{a}(\nu)$ and so can be viewed as a function on $\mathfrak{F}_\mathfrak{a}(\nu)$. Using the fact that $\varphi_\nu(\mathfrak{F}_\mathfrak{a}(\nu))$, for integral $\mathfrak{a}$, is contained in the so-called Siegel domain in $\tilde{H}_\nu$ (Proposition 4), we can prove that the integral of $f_\mathfrak{a}$, with the modified factor $y^{-3\nu(\nu+1)/8}$, on $\mathfrak{F}_\mathfrak{a}(\nu)$ with respect to the invariant volume element of H is convergent (Theorem 1). Though our integral does not give a direct analogy to Siegel-Eisenstein's formula, it is conjectured that in finding the true nature of the value of this