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On Dirichlet series whose coefficients are class numbers of integral binary cubic forms

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Introduction

0.1. We call, after M. Sato, a pair (G, V) of a finite dimensional complex vector space V and an algebraic subgroup G in $GL(V)$ a *prehomogeneous vector space* when there exists a G-orbit in V of the same dimension as V. M. Sato constructed a systematic theory of prehomogeneous vector spaces, and as an application of his results, attached certain "distribution valued zeta-functions" to prehomogeneous vector spaces satisfying several additional conditions.[1] It was also pointed out by Sato that there would exist certain Dirichlet series with functional equations which are intimately related to them, when G is defined over an algebraic number field. In the present paper we give an example of such Dirichlet series whose coefficients have arithmetical significance.

0.2. To state the main result in an explicit form, let L denote the lattice of integral binary cubic forms:

$$L=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3\,;\ (x_1, x_2, x_3, x_4)\in \boldsymbol{Z}^4\}\,.$$

The lattice L becomes an $SL(2, \boldsymbol{Z})$-module if we put

$$\gamma\cdot F(u, v)=F((u, v)\gamma) \qquad (\gamma\in SL(2, \boldsymbol{Z}),\ F\in L)\,.$$

We call two elements x, y of L equivalent if there exists a $\gamma\in SL(2, \boldsymbol{Z})$ such that $x=\gamma\cdot y$. For every integer $m\neq 0$, we denote by L_m the set of integral binary cubic forms whose discriminants are m. It is known that there exist only finitely many equivalence classes in L_m. We denote by $h(m)$ the number of equivalence classes in L_m. We put

$$\hat{L}=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3\in L\,;\ x_2, x_3\in 3\boldsymbol{Z}\}\,.$$

Then $\hat{L}$ is an $SL(2, \boldsymbol{Z})$-submodule of L. We denote by $\hat{h}(m)$ the number of equivalence classes in L_m which are contained in $\hat{L}$. Now we define four Dirichlet series as follows:

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1) A Survey of "the theory of prehomogeneous vector spaces" is given in [8].