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Non-normal functions $f(z)$ with $\iint_{|z|<1} |f'(z)| dxdy < \infty$

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1. Let $f(z)$ be a function holomorphic in the open unit disk D. The spherical derivative of $f(z)$ is given by

$$\rho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

The function $f(z)$ is said to be *normal* in D (see [5]) if there exists a constant $K>0$ such that

$$\rho(f(z)) \leqq \frac{K}{1-|z|^2}$$

for each $z \in D$; and $f(z)$ is said to be *uniformly normal* in D (see [2]) if there exists a constant $K>0$ such that

$$|f'(z)| \leqq \frac{K}{1-|z|^2}$$

for each $z \in D$.

Using the notations

$$\mathcal{D}(f) = \iint_D |f'(z)|^2 dxdy$$

and

$$\mathcal{S}(f) = \iint_D |f'(z)| dxdy,$$

we state the following questions:

(1) Does $\mathcal{D}(f) < \infty$ imply $f(z)$ is uniformly normal?

(2) Does $f(z)$ uniformly normal imply $\mathcal{D}(f) < \infty$?

(3) Does $\mathcal{S}(f) < \infty$ imply $f(z)$ is uniformly normal?

(4) Does $f(z)$ uniformly normal imply $\mathcal{S}(f) < \infty$?

Mathews [6] has answered question (1) in the affirmative; and questions (2) and (4) have been answered in the negative by Mergeljan [7] who has proved the existence of a bounded holomorphic function $g(z)$ for which

$$\iint_D |g'(z)| dxdy = \infty.$$