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Structure of rings satisfying certain polynomial identities

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A well-known theorem of Jacobson [2] asserts that if R is an associative ring with the property that, for all x in R, there exists an integer $m(x)>1$ such that $x^{m(x)}=x$, then R is isomorphic to a subdirect sum of fields. Our present object is to extend Jacobson's Theorem by determining the structure of a certain class of associative rings satisfying polynomial identities involving n elements $x_1, \cdots, x_n$ of R. In order to be able to state this generalization, we first define a *word* $w(x_1, \cdots, x_n)$ in $x_1, \cdots, x_n$ to be a product in which each factor is x_i for some $i=1, \cdots, n$. A *polynomial* $f(x_1, \cdots, x_n)$ is, then, an expression of the form $c_1w_1(x_1, \cdots, x_n)+\cdots+c_mw_m(x_1, \cdots, x_n)$, where the c_i are integers. The *degree* of x_i in the word $w(x_1, \cdots, x_n)$ is the number of times x_i appears as a factor in $w(x_1, \cdots, x_n)$. Suppose that $f(x_1, \cdots, x_n)=c_1w_1(x_1, \cdots, x_n)+\cdots+c_mw_m(x_1, \cdots, x_n)$ is a polynomial in $x_1, \cdots, x_n$. The *degree* of x_i in $f(x_1, \cdots, x_n)$ is the *smallest* value among the following: degree of x_i in $w_1(x_1, \cdots, x_n), \cdots$, degree of x_i in $w_m(x_1, \cdots, x_n)$. The following theorem is proved:

THEOREM 1. *Suppose R is an associative ring and n is a fixed positive integer. Suppose that for all elements $x_1, \cdots, x_n$ of R, there exists a polynomial $f=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$, depending on $x_1, \cdots, x_n$, such that degree of each x_i in $f \geqq 2$, and suppose*

$$x_1 \cdots x_n = f_{x_1,\cdots,x_n}(x_1, \cdots, x_n).$$

Then R is isomorphic to a subdirect sum of fields and a nilpotent ring S satisfying $S^n=(0)$.

Observe that Theorem 1 generalizes Jacobson's Theorem quoted above (take $n=1$ and $f_{x_1}(x_1)=x_1^{m(x_1)}$).

In preparation for the proof of Theorem 1, we proceed to establish the following lemmas. But, first, *we make the assumption that $n>1$ throughout*, since Theorem 1 is true for $n=1$ (see proof of Lemma 3).

LEMMA 1. *Suppose S is an associative subdirectly irreducible ring which does not have an identity. Suppose, moreover, that for all $x_1, \cdots, x_n$ in S, there exists a polynomial $f=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$, depending on $x_1, \cdots, x_n$ such that*

(1) $$x_1 \cdots x_n = f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\text{; degree of each } x_i \text{ in } f \geqq 2.$$