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Herbrand uniformity theorems for infinitary languages[1)]

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We begin by recalling several aspects of Herbrand's theorem for $L_{\omega,\omega}$, or more precisely, of several corollaries to Herbrand's original theorem ([**3**], [**6**], [**8**], not in these references but elsewhere in the literature these corollaries are sometimes confused with the theorem itself). $L^{\vee}$ is an extension of L by arbitrarily many function symbols of each number of arguments.

(1) *Semantic versions.*

(a) *Reduction, for validity, to existential sentences.* For every sentence φ of L there is an existential sentence $\check{\varphi}$ of $L^{\vee}$ such that φ is valid if and only if $\check{\varphi}$ is valid.

(b) *Weak Uniformity theorem.* A prenex existential sentence $\theta=\exists x_1 \cdots x_m \phi(x_1, \cdots, x_m)$ is valid if and only if it is valid in all canonical (term) models; i.e., if and only if for each model $\mathfrak{A}$ of θ there are terms $t_1, \cdots, t_m$ such that $\mathfrak{A} \models \phi(t_1, \cdots, t_m)$.

(b)′ *Uniformity theorem.* θ is valid if and only if for some finite set T of terms $\bigvee\!\!\!\bigvee_{t_1,\cdots,t_m \in T} \phi(t_1, \cdots, t_m)$ is valid.

A third aspect of Herbrand's theorem will be considered in (2) (b) below.

There are many possible sentences $\check{\varphi}$ which can be used for a given φ in (1)(a). In the case that φ is in prenex form, the *validity functional form* (often called the Herbrand normal form), which is dual to the Skolem form, always suffices. For example, if $\varphi=\exists y \forall z \varphi_1(y, z)$ with φ_1 quantifier-free, the validity functional form of φ is

(i) $$\exists y \varphi_1(y, f(y)) .$$

Following Denton and Dreben [**3**], one can directly associate existential $\check{\varphi}$ with *any* φ; $\check{\varphi}$ is an Herbrand normal form of a prenex form of φ.

To illustrate (1)(a) and (1)(b)′, consider the sentence

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