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A note on the vanishing of certain 'L^2-cohomologies'

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Introduction

Let G be a connected noncompact semisimple Lie group admitting a finite dimensional faithful representation. Let K be a maximal compact subgroup of G. Throughout, we assume that G/K is a hermitian symmetric space. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. Then, as is well known, rank of $\mathfrak{k}$ = rank of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. For an ordering of the roots Σ of $(\mathfrak{h}^C, \mathfrak{g}^C)$ compatible with the complex structure on G/K, let P be the set of positive roots and P_k the set of positive compact roots. Let ρ be half the sum of the roots in P. Let $\mathcal{F}$ be the set of all integral linear forms on $\mathfrak{h}^C$. Let

$$\mathcal{F}' = \{\lambda \in \mathcal{F} \mid \langle \lambda+\rho, \alpha \rangle \neq 0, \text{ for } \alpha \in P\}$$

and

$$\mathcal{F}_0' = \{\lambda \in \mathcal{F}' \mid \langle \lambda+\rho, \alpha \rangle > 0, \text{ for } \alpha \in P_k\}.$$

For $\lambda \in \mathcal{F}_0'$, let τ_λ be the irreducible unitary representation of K with highest weight λ on a vector space V_λ. Let τ_λ^* be the contragredient representation of K on the dual V_λ^* to V_λ and let $E_{V_\lambda^*}$ be the holomorphic vector bundle on G/K associated to τ_λ^*. Let $H_2^{0,q}(E_{V_\lambda^*})$ be the Hilbert space of square integrable harmonic forms of type $(0, q)$ with coefficients in $E_{V_\lambda^*}$ and let π_λ^q be the unitary representation of G on $H_2^{0,q}(E_{V_\lambda^*})$. If $\lambda+\rho$ is "sufficiently regular" it was proved in [5, Theorem 2, § 7] that $H_2^{0,q}(E_{V_\lambda^*})=0$, if $q \neq q_\lambda$, where q_λ is the number of non-compact positive roots α such that $\langle \lambda+\rho, \alpha \rangle > 0$ and that $[\pi_\lambda^{q_\lambda}] = \omega(\lambda+\rho)^*$ where $[\pi_\lambda^{q_\lambda}]$ denotes the equivalence class of the representation $\pi_\lambda^{q_\lambda}$ and $\omega(\lambda+\rho)^*$ is the discrete class contragredient to the discrete class $\omega(\lambda+\rho)$ which corresponds to λ (the correspondence being in the sense of Lemma 2.4 in [5]).

Define for $\lambda \in \mathcal{F}_0'$,

$$P^{(\lambda)} = \{\alpha \in \Sigma \mid \langle \lambda+\rho, \alpha \rangle > 0\}.$$

$P^{(\lambda)}$ is the set of positive roots with respect to a linear order in Σ. The main theorem (Theorem 1, § 1) of this note is that if every noncompact root in $P^{(\lambda)}$ is totally positive (in the sense of definition, p. 752 in [2.b]) in the