J. Math. Soc. Japan
Vol. 23, No. 4, 1971

A remark on the character ring of a compact Lie group

By Masaru TAKEUCHI

(Received March 29, 1971)

Introduction

Let G be a compact topological group, $D(G)$ the set of equivalence classes of irreducible representations of G. (In this note the representation will mean always the continuous complex representation.) The character ring $R(G)$ of G is the free abelian group generated by $D(G)$ with the ring structure induced by the tensor product of representations. In the present note we provide a method of finding a system of generators of the character ring $R(G)$ of a compact (not necessarily connected) Lie group G, assuming that the quotient group G/G_0 of G modulo the connected component G_0 of G is a cyclic group (Theorem 5). Our problem reduces to finding generators of a certain commutative semi-group in the similar way as for a compact *connected* Lie group.

By applying the theorem we can know the structure of the character ring of the orthogonal group $O(2l)$ of degree $2l$ or of the double covering group $\mathrm{Pin}(2l)$ of $O(2l)$. (See § 3 for the definition of $\mathrm{Pin}(2l)$.) Let λ^i be the i-th exterior power of the standard representation of $O(2l)$, α the 1-dimensional representation of $O(2l)$ defined by $\alpha(x) = \det x$ for $x \in O(2l)$. Let μ^l be the irreducible representation of $\mathrm{Pin}(2l)$ such that its restriction to the connected component $\mathrm{Spin}(2l)$ of $\mathrm{Pin}(2l)$ splits into the direct sum of two half-spinor representations of $\mathrm{Spin}(2l)$ and $p: \mathrm{Pin}(2l) \to O(2l)$ denote the covering homomorphism. Then we have

$$R(O(2l)) = \boldsymbol{Z}[\lambda^1, \lambda^2, \cdots, \lambda^l, \alpha] \text{ with relations } \alpha^2 = 1 \text{ and } \lambda^l \alpha = \lambda^l,$$

$$R(\mathrm{Pin}(2l)) = \boldsymbol{Z}[\lambda^1 \circ p, \lambda^2 \circ p, \cdots, \lambda^{l-1} \circ p, \mu^l, \alpha \circ p]$$

$$\text{with relations } (\alpha \circ p)^2 = 1 \text{ and } \mu^l(\alpha \circ p) = \mu^l.$$

The character ring of $O(2l)$ was formerly presented by Minami [7] by different methods.