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On the union of two Helson sets

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The purpose of this paper is to improve and generalize some results of N. Th. Varopoulos [8]. In particular, we shall show that the union of two Helson sets in a locally compact abelian group is a Helson set.

We begin with introducing some notations. Let K be any non-empty space, and let Fag (K) be the free abelian (additive) group generated by K with the discrete topology (cf. [3; p. 8]). For any positive integer $l \in Z^+$, we denote

$$K^{(l)} = \{\sum_{i=1}^{l} n_i x_i\,;\ n_i \in Z,\ x_i \in K,\ \sum_{i=1}^{l} |n_i| \leqq l\}\,,$$

which is a subset of Fag (K). Let also $F^*(K)$ be the multiplicative group consisting of all complex-valued functions f on K such that $|f(x)|=1$ for all $x \in K$. $F^*(K)$ is a metric abelian group under the metric

$$d(f, g) = \sup_{x \in K} |f(x)-g(x)| \qquad (f, g \in F^*(K))\,.$$

Then it is easy to see that every element x of Fag (K) defines a continuous character of $F^*(K)$ by

$$\langle f, x\rangle = \prod_{i=1}^{l} \{f(x_i)\}^{n_i} \qquad (f \in F^*(K))\,,$$

where $n_i \in Z$ and $x_i \in K$ are such that $x=\sum_{i=1}^{l} n_i x_i$. This fact allows us to identify $F^*(K)$ with a subgroup of $F^*(\mathrm{Fag}\,(K))$.

Suppose now that $D=\{K_j\}_1^N$ be any finite partition of K into pairwise disjoint, non-empty subsets. We denote by $F^*_D = F^*_D(K)$ the closed subgroup of $F^*(K)$ consisting of those functions of $F^*(K)$ that are constant on each set K_j $(j=1, 2, \cdots, N)$. It is trivial that F^*_D is topologically isomorphic to the N-dimensional torus $T^N=\{z\,;\ |z|=1\}^N$. Let now p be a given, continuous, positive-definite function on F^*_D, and let $\{x_j \in K_j\}_1^N$ be any choice of points. We can identify the subgroup of Fag (K)

$$G_p(\{x_j\}_1^N) = \{\sum_{j=1}^{N} n_j x_j\,;\ n_j \in Z,\ j=1, 2, \cdots, N\}$$

with the dual of F^*_D in a trivial way. It follows from the classical Bochner