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Free Hopf algebras generated by coalgebras

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Introduction

If H is a commutative or cocommutative Hopf algebras over a field k, it is well known that the antipode of H is of order 2. If H is a finite dimensional Hopf algebra over k, then the antipode of H is a bijection. Is the antipode of a Hopf algebra always a bijection? In this paper we construct some Hopf algebras whose antipodes are not bijective. In order to do so, we introduce the concept of free Hopf algebras generated by coalgebras.

Let C be a coalgebra over a field k. The free Hopf algebra $(H(C), i)$ generated by C is characterized by the following universal property:

(1) $i: C \to H(C)$ is a coalgebra map

(2) $\mathrm{Hom}(i, H): \mathrm{Hopf}(H(C), H) \to \mathrm{Coalg}(C, H)$ is a bijection for any Hopf algebra H.

$H(C)$ is constructed in § 1. One of our main results is the following

THEOREM. *The antipode of $H(C)$ is bijective if and only if the $\bar{k}$-coalgebra $\bar{k} \otimes C$ is pointed, where $\bar{k}$ is the algebraic closure of k.*

Some important consequences are obtained as corollaries to this theorem.

It is interesting to consider the algebra structure of $H(C)$ and to give a k-basis for $H(C)$ explicitly. We present a partial answer to this problem in Chapter III. In Chapter IV we consider the corresponding problem in the category of commutative algebras. The definition of the free commutative Hopf algebra $H_c(C)$ generated by a coalgebra C is similar to that of $H(C)$. In the category of commutative algebras, there is an interesting relation between norms and antipodes. This relation leads to a simple construction of $H_c(C)$. A consequence of this construction is

THEOREM. *A commutative bialgebra H has an antipode if and only if the grouplike elements of H are invertible in H.*

This is a generalization of [5, Prop. 9.2.5] in the category of commutative algebras.

Throughout this paper, we shall adopt the terminology and utilize theorems in [5].