J. Math. Soc. Japan
Vol. 23, No. 4, 1971

On skew product transformations with quasi-discrete spectrum

By Nobuo AOKI

(Received Dec. 1, 1969)
(Revised May 4, 1971)

§ 1. Introduction.

Let X and Y be unit intervals with Borel measurability and Lebesgue measure. Let $\Omega=X\otimes Y$ be the unit square with the usual direct product measurability and measure. We consider the following skew product (measure preserving) transformation defined on Ω; let T be the measure preserving transformation with the α-function defined by $T:(x, y)\rightarrow(x+\gamma, y+\alpha(x))$ (additions modulo 1) where γ is an irrational number and $\alpha(\cdot)$ a real-valued measurable function defined on X.

The purpose of this paper is to give a criterion in order that the transformation T has quasi-discrete spectrum.

I am greatly indebted to the referee for many improvements on this paper.

§ 2. Definitions.

Let (Z, Σ, m) be a finite measure space and T an invertible measure preserving transformation on Z. We recall the following definition of quasi-proper functions [1]. Let $G(T)_0$ be the set

$$\{\beta\in K:\ V_T f=\beta f,\ \|f\|_2=1 \text{ for } f\in L^2(Z)\},$$

where V_T is the unitary operator induced by the transformation T and K the unit circle in the complex plane. For each positive integer i, let $G(T)_i\subset L^2(Z)$ be the set of all normalized functions f such that $V_T f=gf$ where $g\in G(T)_{i-1}$. The set $G(T)_i$ is the set of quasi-proper functions of order at most i. We put $G(T)=\bigcup_{i\geqq 0}G(T)_i$. The transformation T is said to have *quasi-discrete spectrum* if the set $G(T)$ spans $L^2(Z)$. If the set $G(T)_1$ of order 1 spans $L^2(Z)$, then it is well-known that T has discrete spectrum. If the transformation T is ergodic, then $|f(x)|=1$ for arbitrary $f\in G(T)$. This implies that $G(T)$ is a

---

Throughout this paper, any equality between functions are taken as the equality for almost all values of the variables.