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On realization of the discrete series for semisimple Lie groups

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§ 0. Introduction.

The main purpose of this paper is to show that most of the discrete series for a semisimple Lie group are realized on certain eigenspaces of the Casimir operator over the symmetric space. In more detail, let G be a connected non-compact semisimple Lie group with a finite dimensional faithful representation and K a maximal compact subgroup of G. Assume that $\mathrm{rank}\, G = \mathrm{rank}\, K$ (according to [**6**, Theorem 13], G has a discrete series if and only if G satisfies this condition). Let V_λ be an irreducible unitary K-module with lowest weight $\lambda+2\rho_k$, where ρ_k is the half sum of positive compact roots. We denote by $C^\infty(\mathcal{V}_\lambda)$ (resp. $L_2(\mathcal{V}_\lambda)$) the space consisting of all V_λ-valued C^∞ (resp. square-integrable) functions f on G such that $f(gk)=k^{-1}f(g)$ for $g\in G$, $k\in K$. Denoting by Ω the Casimir operator of G, let Ω act on $C^\infty(\mathcal{V}_\lambda)$ in the usual manner and denote by $\nu(\Omega)$ the differential operator given by the action of Ω on $C^\infty(\mathcal{V}_\lambda)$ in this sense (for a precise definition, see § 1). Put

$$\mathfrak{H}_\lambda=\{f\in C^\infty(\mathcal{V}_\lambda)\cap L_2(\mathcal{V}_\lambda)\,;\ \nu(\Omega)f=\langle\lambda+2\rho,\lambda\rangle f\}\,,$$

where ρ denotes the half sum of all positive roots and $\langle\ ,\ \rangle$ denotes the usual inner product on the set of weights induced by the Killing form. Since $\nu(\Omega)$ is elliptic on $C^\infty(\mathcal{V}_\lambda)$, $\mathfrak{H}_\lambda$ is then a Hilbert space and gives a unitary representation of G through the left translation. Assume that $\langle\lambda+\rho,\alpha\rangle<0$ for all positive roots α. Then, there exists a constant a such that if $|\langle\lambda+\rho,\beta\rangle| > a$ for all non-compact positive roots β, $\mathfrak{H}_\lambda$ gives an irreducible unitary representation belonging to the discrete series for G, which is equivalent to the discrete class $\omega(\lambda+\rho)$ in the sense of [**6**] (§3, Corollary to Theorem 2). In view of Harish-Chandra's result [**5**], [**6**], the above result gives a procedure in order to realize most of the discrete series for G.

For our proof, we make use of the method established by M. S. Narasimhan and K. Okamoto in [**11**]. That is, the above result is deduced from Theorem 1 in § 2 and Lemma 9 in § 3, which amount to generalizations of the alternating sum formula and the vanishing theorem [**11**, Theorem 1 and Theorem 2] respectively.