J. Math. Soc. Japan
Vol. 23, No. 1, 1971

# On Toeplitz operators

By Matthew C. Y. LEE

(Received May 19, 1970)

Let $L^2$ and $L^\infty$ denote the Lebesgue spaces of square integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let $H^2$ and $H^\infty$ denote the corresponding Hardy spaces. For $\phi$ in $L^\infty$, *the Toeplitz operator induced by* $\phi$ is the operator $T_\phi$ on $H^2$ defined by $T_\phi f = P(\phi f)$; here $P$ stands for the orthogonal projection in $L^2$ with range $H^2$.

The purpose of this paper is to prove an inversion theorem (Theorem 2) of $T_f$ for $f$ in a class of subalgebras $A_\phi$ of $H^\infty + C$, and then we can determine (Theorem 3) the spectrum of $T_f$, for any unitary function $f$ in $A_\phi$. We recall that the linear span $H^\infty + C$ of $H^\infty$ and $C$ is a closed subalgebra of $L^\infty$ [**4**, Theorem 2], where $C$ stands for the space of continuous complex valued functions on the unit circle. This algebra can also be characterized as the subalgebra of $L^\infty$ generated by $H^\infty$ and the function $\bar{z}$. Let $\mathcal{B}$ denote the algebra of bounded operators on $H^2$, $\mathcal{K}$ the uniformly closed two-sided ideal of compact operators in $\mathcal{B}$, and $\pi$ the homomorphism of $\mathcal{B}$ onto $\mathcal{B}/\mathcal{K}$. An operator $B$ in $\mathcal{B}$ is said to be a Fredholm operator if $B$ has a closed range and both a finite dimensional kernel and cokernel. It is known [**1**] that this is equivalent to $\pi(B)$ being an invertible element of $\mathcal{B}/\mathcal{K}$. If $B$ is a Fredholm operator, then the index $\mathrm{ind}(B)$ is defined $\mathrm{ind}(B) = \dim[\ker B] - \dim[\operatorname{coker} B]$. In general for a Fredholm operator $B$ the statement $\mathrm{ind}(B) = 0$ does not imply that $B$ is invertible. For Toeplitz operators, however, the situation is simpler as was shown by Coburn [**2**].

LEMMA 1. *If $\phi$ is in $L^\infty$ such that $T_\phi$ is a Fredholm operator and* $\mathrm{ind}(T_\phi) = 0$, *then $T_\phi$ is invertible.*

Stampfli observed in [**5**] that $T_\phi T_z - T_z T_\phi$ is at most one dimensional for any $\phi$ in $L^\infty$ and hence compact. Therefore, $T_f T_g - T_g T_f$ is a compact operator for any $f$ and $g$ in $H^\infty + C$ and $T_f T_\phi - T_\phi T_f$ is a compact operator for any $\phi$ in $L^\infty$ if $f$ is in $C$.

LEMMA 2. *Let $f$ be in $H^\infty + C$, then $T_h T_f - T_{hf}$ is a compact operator for every $h$ in $L^\infty$.*

PROOF. Since $f$ is in $H^\infty + C$, we can write $f = f_1 + f_2$ where $f_1$ in $H^\infty$ and $f_2$ in $C$. Consider