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Characterization of the simple components of the group algebras over the p-adic number field

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§ 1. Introduction.

Let G be a finite group and K a field of characteristic 0. Then the group algebra $K[G]$ of G with respect to K is semisimple. We can write it as a direct sum

$$K[G]=A_1 \oplus A_2 \oplus \cdots \oplus A_r$$

of simple algebras. Each A_i is in one-to-one correspondence with a family T_i of absolutely irreducible characters $\chi_{i\nu}(\nu=1,\cdots,t_i)$ of G, taken in the algebraic closure $\bar{K}$ of K and algebraically conjugate to each other over K. Each simple algebra A_i is isomorphic to a complete matrix algebra $M_{\rho_i}(\Delta_i)$ of a certain degree ρ_i with coefficients in a division algebra Δ_i over K. Let $K(\chi_{i\nu})$ denote the field obtained from K by adjoining all values $\chi_{i\nu}(g)$ with $g \in G$ of the character $\chi_{i\nu}$. It turns out that the center Ω_i of Δ_i is isomorphic to $K(\chi_{i\nu})$ for $\chi_{i\nu} \in T_i$. If the dimension of Δ_i over Ω_i is m_i^2, m_i is called the Schur index of the division algebra Δ_i or of the characters $\chi_{i\nu}$ $(\nu=1,\cdots,t_i)$.

Now we are faced with the problem: Characterize division algebras which appear at simple components of group algebras.

In this paper this problem is solved for division algebras over the p-adic number field $\boldsymbol{Q}_p$, where p is any odd prime number. Namely, we shall prove the following

THEOREM 1. *Let p be an odd prime number. Denote by Ξ the field obtained from $\boldsymbol{Q}_p$ by adjoining all primitive roots of unity ζ_n $(n=3,4,5,\cdots)$. Then, a given (finite dimensional) division algebra Δ over $\boldsymbol{Q}_p$ appears at a simple component of the group algebra $\boldsymbol{Q}_p[G]$ over $\boldsymbol{Q}_p$ of a certain finite group G if and only if* (i) *the center k of Δ is a finite extension field of $\boldsymbol{Q}_p$ contained in Ξ, and* (ii) *the Hasse invariant of Δ is of the form*

$$z/\frac{p-1}{b} \pmod{\boldsymbol{Z}}, \qquad z \in \boldsymbol{Z},$$

where $\boldsymbol{Z}$ is the ring of rational integers and b is the index of tame ramification

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