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Spectrum of a substitution minimal set

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§ 1. Summary

K. Jacobs ([1]) reported as an example of Toeplitz type sequences that

0 0 0 0 0 0 0 0 0 0 0 0 0 ⋯

1   1   1   1   1   1   ⋯

0   0   0   ⋯

1   1 ⋯

0   ⋯

= 0100010101000100010001010 ⋯

is strictly ergodic and has a rational pure point spectrum. This sequence has the following properties:

(i) It is a shift of the sequence 001000101010001⋯ which is invariant under the substitution $0 \to 0010$, $1 \to 1010$ of length 4.

(ii) The $(2i+1)$-th symbol of it is 0 for $i = 0, 1, 2, \cdots$.

In this paper, we prove that if some general conditions like (i) (ii) above are satisfied for a sequence over some finite alphabet, then it is strictly ergodic and has a rational pure point spectrum. That is, our main results are the followings:

I. If M is a minimal set associated with a substitution of some constant length, then M is strictly ergodic.

II. Let M be a strictly ergodic set associated with a substitution of length p^k, where p is a prime number and k is any positive integer. Assume that for some (or, equivalently, any) $\alpha \in M$, there exist integers $h \geqq 0$ and $r \geqq 1$, such that $(ip^h + r)$-th symbol of α is the same for $i = 0, 1, 2, \cdots$. Then, M has a rational pure point spectrum $\{\omega ; \omega^{p^i} = 1 \text{ for some } i = 0, 1, 2, \cdots\}$.

§ 2. Notations and definitions

Let C be any finite set of symbols which contains at least two elements. Let $N = \{0, 1, 2, \cdots\}$ be the set of non-negative integers. Let T be the shift transformation on the power space C^N. That is, T is defined as follows: